HOW TO DE-RESERVE RESERVES

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Abstract

Reserve systems have been designed and implemented for numerous real-world resource allocation problems. Often, *de-reservation policies* accompany reserve systems to prevent waste in instances of low demand for *exclusive* reserve categories. De-reservation policies must be executed carefully so that allocation mechanisms have desired properties. We evaluate the de-reservation policy that has been implemented in admissions to technical universities in India and reveal its shortcomings. We introduce two families of choice procedures—*backward* and *forward transfers choice rules*—and deferred acceptance (DA) mechanisms with respect to these choice rules to retrieve these shortcomings. We introduce a framework to compare choice rules on the basis of merit and show that forward transfers choice rules select more meritorious sets than backward transfers choice rules. We further compare the DA mechanisms under backward and forward transfers choice rules on the basis of merit and individuals' welfare.

Keywords : Reserve Systems, De-reservation, Affirmative Action, Market Design.

JEL Codes*:* C78, D47, D63.

1 Introduction

Reserve systems have been designed and implemented to allocate scarce resources in the presence of diversity and affirmative action considerations in a variety of real-world problems. Notable examples include:

- allocation of publicly funded school seats and government-sponsored job positions in India (Baswana et al. 2018 and 2019; Aygün and Turhan 2017, 2020, and 2021; Thakur 2018; Sönmez and Yenmez 2020 and 2021),
- college admission in Brazil with multidimensional privileges (Aygün and Bó 2020),
- school choice in Boston and Chile (Dur et al. 2018; Correa et al. 2019),
- immigration visas in the US (Pathak et al. 2020),
- allocation of vaccines, ventilators, and other medical resources (Pathak et al. 2020; Aziz and Brandl 2021), and
- Mechinot gap-year programs in Israel (Gonczarowski et al. 2020).

In reserve systems, certain fractions of available objects/positions are set aside for different *reserve categories*. Each reserve category has its own priority order over individuals. Institutions process reserve categories *sequentially*[1] to fill their positions, according to a pre-specified order referred to as a *processing sequence*. Reserve categories allocate their units to the individuals, who have not yet been allotted a unit on the basis of their priorities. Priorities may vary from one reserve category to another to accommodate affirmative action constraints or to promote diversity, among other objectives.

In almost all real-world applications, most reserve categories are *exclusive* in the sense that only applicants with certain types or characteristics are considered. That is, if positions in a reserve category can only be allocated to individuals with a certain characteristic, then all other individuals who do not possess this characteristic are deemed unqualified and are unacceptable according to the priority ordering of this reserve category.[2] It is highly common for the number of available positions to outnumber the number of applicants of such

[1] Delacretaz (2020) formulates a model where categories allocate their unit/positions simultaneously.

[2] In vaccine allocation during COVID-19, for example, frontline healthcare workers and people who live in care homes are considered as exclusive reserve categories. In Brazilian college admission, low-income minority students from public high schools are an example of an exclusive reserve category. In government-sponsored job allocation in India, candidates from Scheduled Castes are an example of an exclusive reserve category.

exclusive reserve categories. Therefore, objects may be unassigned or positions may be unfilled in such exclusive reserve categories.[3] To alleviate this issue, *de-reservation* policies have been introduced along with accompanying reservation policies. De-reservation is simply a process of providing the unallocated objects/positions for the use of others. It can be interpreted as transferring units or positions from low-demand reserve categories to high-demand ones.

The lack of de-reservation policies may cause confusion, and even have legal consequences. Examples have been seen during the COVID-19 vaccine allocation in many countries. Almost all countries implement a reserve system to allocate vaccines, starting with vaccinating frontline healthcare workers followed by elderly in care homes. Most countries did not specify de-reservation policies for when they have leftover doses. In Austria, for example, the government did not provide guidelines for handling leftover doses before vaccine distribution started. Local authorities allocated leftover vaccines according to their own judgments and have been accused of jumping the queue, some have faced legal challenges.[4]

De-reservation policies are necessary in many real-world allocation problems. Moreover, the implementation of de-reservation policies is consequential. When de-reservation policy is not designed and/or implemented carefully, allocation procedures as a whole might have serious shortcomings, no matter how well-designed the reserve system is. Reserve policies have been well-studied in the literature, while de-reservation policies that are attached to reserve systems have not. This paper aims to fill this crucial gap. As we will show, the design and implementation of de-reservation policies affects whether the reserve systems and affirmative action programs can yield the full benefit.

The organization of the paper is as follows: In Section 2, we model the admissions market of technical universities in India and formulate the currently implemented sequential mechanism—*multi-run deferred acceptance*—to handle de-reservations. In the same section, we disclose its shortcomings. In Section 3, we introduce two families of choice rules—*backward* and *forward transfers choice rules*—to untangle de-reservations and the *deferred acceptance mechanism* with respect to these choice rules. We show that our proposals can successfully overcome the shortcomings of the multi-run deferred acceptance and satisfactorily clear the market. In Section 4, we compare backward and forward trans-

[3]This has been happening in the allocation of publicly funded school seats in India.

[4]The news article can be accessed at https://www.theguardian.com/world/2021/jan/21/austrian-mayors-who-got-leftover-covid-vaccines-accused-of-queue-jumping (last accessed on 01/23/2021).

fers choice rules with respect to a comparison criteria on the basis of merit, and with respect to individuals' welfare. In the same section, we extend these comparisons to the outcomes of the DA with respect to these choice rules. Section 5 discusses the related literature and Section 6 concludes.

2 Framework

2.1 Admission Market of Technical Universities in India

The admission process at technical universities in India functions through a centralized marketplace that matches approximately 1.3 million students to 34,000 university seats. The process was recently reformed and the new procedure has been adopted beginning in 2015. The reform was the product of collaboration between policymakers and researchers from computer science and operations research, and was summarized in Baswana et al. (2018). The authors report the design and implementation of the new procedure, which is based on the celebrated deferred acceptance algorithm of Gale and Shapley (1962).

Admissions to publicly funded universities in India are subject to an affirmative action program that has been implemented via a reservation policy for decades. According to the reservation policy, each institution sets aside 15 percent of its slots for applicants from *Scheduled Castes* (SC), 7.5 percent for applicants from *Scheduled Tribes* (ST), and 27 percent for applicants from *Other Backward Classes* (OBC). Applicants who do not belong to any of these categories are referred to as members of the *General Category* (GC). The remaining slots are called *open-category* slots and are available to everyone, including applicants from SC, ST, and OBC. In each institution, for slots that are reserved for SC, ST, and OBC, only applicants who declare they belong to these respective categories are considered. In each institution, open-category positions are filled first, followed by the exclusive reserve categories. The processing order of seat categories for different applicant types is as follows:

- Applicants who declare their SC memberships are first considered for open-category positions, and then for reserved SC positions,
- Applicants who declare their ST memberships are first considered for open-category positions, and then for reserved ST positions,

- Applicants who declare their OBC memberships are first considered for open-category positions, and then for reserved OBC positions,

- Applicants who do not declare membership to SC, ST, or OBC are only considered for open-category positions.

By law, vacant SC/ST positions cannot be de-reserved. By and large, many SC/ST positions remain vacant and are wasted each year. On the other hand, unfilled OBC positions must be de-reserved. If there are not sufficient OBC applicants, the unfilled OBC positions are converted into open-category positions. To model the reserve system and the de-reservation policy and how they are implemented, we will first formulate the technical university admissions market.

There is a finite set of institutions (programs) $\mathcal{S} = \{s_1, ..., s_m\}$ and a finite set of individuals $\mathcal{I} = \{i_1, ..., i_n\}$. We denote the number of available positions at institution $s \in \mathcal{S}$ by $\overline{q}_s$. For each institution s, the vector $\left(q_s^{SC}, q_s^{ST}, q_s^{OBC}\right)$ denotes the number of positions reserved for SC, ST, and OBC categories. We let $\mathcal{R} = \{SC, ST, OBC\}$ to denote the set of reserve categories, and let $C = \{OP, SC, ST, OBC\}$ to denote the set of all categories[5]. The number of open-category seats at institution s is $q_s^{OP} = \overline{q}_s - q_s^{SC} - q_s^{ST} - q_s^{OBC}$. We write $q_s = \left(q_s^{OP}, q_s^{SC}, q_s^{ST}, q_s^{OBC}\right)$ to describe the initial distribution of positions over reserve categories at institution s. Let $\mathbf{q} = (q_s)_{s \in \mathcal{S}}$ denote a profile of vectors for the initial distribution of positions over categories (OP, SC, ST,OBC) at institutions. That is, $\mathbf{q}$ is a vector of distribution vectors.

The function $\mathbf{t} : \mathcal{I} \to C$ denotes the category membership of individuals. For every individual $i \in \mathcal{I}$, $\mathbf{t}(i) \in C$ denotes the category individual i belongs to. In India, it is **optional** to report SC, ST, or OBC membership. Reserved category members who do not report their membership are considered GC applicants and are eligible only for open-category positions. Members of reserve category $r \in \mathcal{R}$ are eligible for both open-category positions and reserved category r positions. We denote a profile of reserved category membership by $\mathbf{T} = (\mathbf{t}(i))_{i \in \mathcal{I}}$. Let $\mathcal{T}$ be the set of all possible reserved category membership profiles.

The function $\kappa : \mathcal{I} \times \mathcal{S} \to \mathbb{R}_+$ denotes individuals' merit scores at institutions. Applicants might have different merit scores for different institutions. We let $\kappa(i, s)$ denote the

[5]In Baswana et al. (2018), there are special reservations for People with Disabilities (PwD) within each of these categories. Namely, SC-PwD, ST-PwD, OBC-PwD, and GC-PwD. For the sake of simplicity, we did not model these. Our model can be straightforwardly extended to a model that also captures these special categories. All of our results are independent of this simplification and hold in a model that covers special reservations.

merit score of individual i at institution s. We assume that no two individuals have the same score at a given institution[6]. That is, for all $i, j \in \mathcal{I}$ and $s \in \mathcal{S}$ such that $i \neq j$, we have $\kappa(i,s) \neq \kappa(j,s)$. Merit scores induce strict meritorious ranking of individuals at institution s, denoted $\succ_s$, which is a linear order over $\mathcal{I} \cup \{\emptyset\}$. $i \succ_s j$ means that applicant i has a higher priority (higher merit score) than applicant j at institution s. That is, $\kappa(i,s) > \kappa(j,s)$. We write $i \succ_s \emptyset$ to say that applicant i is acceptable for institution s. Similarly, we write $\emptyset \succ_s i$ to say that applicant i is unacceptable for institution s. The profile of institutions' merit lists is denoted by $\succ = (\succ_{s_1}, ..., \succ_{s_m})$. For each institution $s \in \mathcal{S}$, the merit ordering for applicants of type $t \in \mathcal{R}$, denoted by $\succ_s^t$, is obtained from $\succ_s$ in a straightforward manner as follows:

- for $i, j \in \mathcal{I}$ such that $t(i) = t$, $t(j) \neq t$, $i \succ_s \emptyset$, and $j \succ_s \emptyset$, we have $i \succ_s^t \emptyset \succ_s^t j$,[7]
- for any other $i, j \in \mathcal{I}$, $i \succ_s^t j$ if and only if $i \succ_s j$.

Each reserve category $t \in \mathcal{R}$ is *exclusive*. That is, all applicants who do not belong to category t become unacceptable. Among the applicants who belong to category t, the ranking $\succ_s$ is preserved.

Each individual $i \in \mathcal{I}$ has a strict preference relation P_i over $\mathcal{S} \cup \{\emptyset\}$, where $\emptyset$ denotes the outside option, i.e., remaining unmatched. We write $sP_i\emptyset$ to mean that institution s is *acceptable* for individual i. Similarly, $\emptyset P_i s$ means institution s is *unacceptable* for individual i. We denote the profile of true individual preferences by $P = (P_i)_{i \in \mathcal{I}}$. We denote by R_i the weak preference relation associated with the strict preference relation P_i, and by $R = (R_i)_{i \in \mathcal{I}}$ as the profile weak preferences.

For each institution $s \in \mathcal{S}$, its selection criterion is summarized by a choice function C_s. A choice function C_s simply selects a subset from any given set of individuals. That is, for any given $I \subseteq \mathcal{I}$, the chosen set $C_s(I)$ is a subset of I, i.e., $C_s(I) \subseteq I$. We now introduce a *choice function with reserves* C_s^{Res} that will be key for the rest of our analysis.

Choice Rule with Reserves C_s^{Res}

Given an initial distribution of positions $q_s = \left(q_s^{OP}, q_s^{SC}, q_s^{ST}, q_s^{OBC}\right)$, a set of applicants $A \subseteq \mathcal{I}$, and a profile reserve category membership $\mathbf{T} = (\mathbf{t}(i))_{i \in A}$ for the members of A, the set of chosen applicants $C_s^{Res}(A, q_s)$, is computed as follows:

[6] In India, when two or more applicants have the same score, ties are broken with some exogenously given objective criteria.

[7] $\emptyset \succ_t^s j$ means applicant j is unacceptable for category t at institution s.

Stage 1. Only open-category seats are considered. Individuals are chosen one at a time following $\succ_s$ up to the capacity q_s^{OP}. Let us call the set of chosen applicants $C_s^{OP}\left(A, q_s^{OP}\right)$.

Stage 2. Among the remaining applicants $A' = A \setminus C_s^{OP}\left(A, q_s^{OP}\right)$, for each reserve category $t \in \mathcal{R}$, applicants are chosen one at a time following $\succ_s^t$ up to the capacity q_s^t. Let us call the set of chosen applicants for reserve category t $C_s^t\left(A', q_s^t\right)$.

Then, $C_s^{Res}(A, q_s)$ is defined as the union of the set of applicants chosen in stages 1 and 2. That is,

$$C_s^{Res}(A, q_s) = C_s^{OP}\left(A, q_s^{OP}\right) \cup \bigcup_{t \in \mathcal{R}} C_s^t\left(A', q_s^t\right).$$

This is a commonly used sequential choice procedure in practice. Note that the chosen set is a function of the vector of initial distribution of positions over categories q_s.

A choice rule determines who is chosen from any given set of individuals when there is a single institution. In centralized marketplaces, there are multiple institutions, each of which has its own selection criteria embodied in its choice rule.

Matching and Stability

A matching specifies, for every institution, the set of individuals who are assigned to that institution. Mathematically, a **matching** μ is a function $\mu : \mathcal{I} \cup \mathcal{S} \rightarrow \mathcal{I} \cup \mathcal{S} \cup \{\emptyset\}$ such that

1. for any individual $i \in \mathcal{I}$, $\mu(i) \in \mathcal{S} \cup \{\emptyset\}$,
2. for any institution $s \in \mathcal{S}$, $\mu(s) \subseteq \mathcal{I}$ such that $\mid \mu(s) \mid \leq q_s$,
3. for any individual $i \in \mathcal{I}$ and institution $s \in \mathcal{S}$, $\mu(i) = s$ if and only if $i \in \mu(s)$.

That is, an individual is either matched with an institution, or the outside option $\emptyset$ and an institution s is matched with a set of individuals that has at most $\overline{q}_s$ individuals. Moreover, an individual i is assigned to institution s if and only if i is in the set of individuals matched with s.

Stability has appeared as one of the most important desiderata in the matching markets. We now give the stability definition with respect to a profile of institutional choice rules $\mathbf{C} = (C_s)_{s \in \mathcal{S}}$.

Definition 1. A matching μ is **stable** with respect to the profile of applicants' preferences $P = (P_i)_{i \in \mathcal{I}}$ and a profile choice rules of institutions $\mathbf{C} = (C_s)_{s \in \mathcal{S}}$ if,

(i) for every individual $i \in \mathcal{I}$, $\mu(i) R_i \emptyset$,

(ii) for every institution $s \in \mathcal{S}$, $C_s(\mu(s)) = \mu(s)$, and

(iii) there is no (i, s) such that $s P_i \mu(i)$ and $i \in C_s(\mu(s) \cup \{i\})$.

If the first requirement (*individual rationality for individuals*) fails, then there is an individual who is assigned to an unacceptable institution. In our context, the second condition (*individual rationality for institutions*) requires that the institutions' selection criteria summarized in their choice rules are respected. If the third condition (*unblockedness*) fails, then there is an alternative matching that an individual and an institution strictly prefers.

Stability depends on how institutions' selection procedures are defined. In India, for example, institutions' selection criteria embody legal requirements, such as satisfying reservation and de-reservation policies and respecting merit scores subject to affirmative action. If each individual applies to only one institution, stability requires that the rules and regulations encoded in institutions' choice rules determine which individuals are selected. Stability proves to be a natural desideratum for an allocation: an individual will only be matched to a less desirable institution if, by following the selection criteria of those institutions, she would not be accepted given the individuals who have been matched to these institutions. Unstable allocations, therefore, might lead to lawsuits from dissatisfied applicants.

Mechanisms

A *mechanism* is a systematic way to map preference and reserve category membership profiles of individuals to matchings, given institutions' choice procedures. Technically, a mechanism φ is a function $\varphi : \mathcal{P} \times \mathcal{T} \to \mathcal{M}$, where $\mathcal{P}$ denotes the set of all preference profiles of individuals, $\mathcal{T}$ denotes the set of all reserve category membership profiles, and $\mathcal{M}$ denotes the set of all matchings, given a profile of institutional choice rules $\mathbf{C} = (C^s)_{s \in \mathcal{S}}$.

A mechanism φ is **stable** if $\varphi(P, \mathbf{T})$ is a stable matching for all pairs $(P, \mathbf{T}) \in \mathcal{P} \times \mathcal{T}$.

One of the main desiderata that also has been key for the success of matching mechanisms in practice is strategy-proofness, according to which submitting the true preferences is a weakly dominant strategy for each individual.

Definition 2. A mechanism φ is **strategy-proof** if for every preference profile P and for every reserve category membership profile $\mathbf{T}$, and for each individual $i \in \mathcal{I}$, there is no $\widetilde{P}_i$, such that $\varphi((\widetilde{P}_i, P_{-i}), \mathbf{T}) P_i \varphi(P, \mathbf{T})$.

Affirmative action policies are designed to increase admission chances of members of reserved categories in the sense that the assignment of a reserve category member when she claims her membership is at least as good as the assignment she would receive without reporting her membership. That is, reporting their membership to reserve categories should not hurt them. Otherwise, the rationale behind the affirmative action policy is violated. This idea was first formulated by Aygün and Bó (2020) in the context of college admission in Brazil with multi-dimensional privileges.

We now formulate this notion for our setting.

Definition 3. A mechanism φ is **incentive compatible for reserve category membership revelation** if, for every preference profile $P \in \mathcal{P}$, there is *no* individual $i \in \mathcal{I}$—who is a member of reserve category $r \in \mathcal{R}$—receives a better assignment by not reporting her membership to r given reserve category membership of other individuals $\mathbf{T}_{-i} = \left(\mathbf{t}_j\right)_{j \in \mathcal{I}\setminus\{i\}}$. That is,

$$\varphi_i\left(P;\left(\mathbf{T}_{-i},\mathbf{t}_i^{'}\right)\right) P_i \varphi_i\left(P;\left(\mathbf{T}_{-i},\mathbf{t}_i\right)\right),$$

where $\mathbf{t}_i^{'} = GC$ and $\mathbf{t}_i = r$.

2.2 Formulation of the Current De-reservation Procedure

Before formulating the currently implemented de-reservation policy, we describe the DA algorithm with respect to choice rules with reserves, which will prove useful for describing the sequential version of DA that is implemented to handle de-reservation policy.

DA Algorithm under Choice Rules with Reserves

Suppose that $\widetilde{P} = \left(\widetilde{P}_i\right)_{i \in \mathcal{I}}$ is the vector of the *reported* preference relations and $\mathbf{T} = (\mathbf{t}_i)_{i \in \mathcal{I}}$ is a a vector of reported reserve category membership of individuals. Given institutions' priority rankings $\succ = (\succ_s)_{s \in \mathcal{S}}$ and the profile $\mathbf{q} = (q_s)_{s \in \mathcal{S}}$—therefore, given the choice function with reserves of each institution $s \in \mathcal{S}$, $C_s^{Res}(\cdot; q_s)$—the outcome of the DA algorithm with respect to the choice rules with reserves defined above is found as follows:

Step 1. Each individual in $\mathcal{I}$ applies to his top choice institution. Let $\mathcal{A}_s^1$ be the set of individuals who apply to institution s, for each $s \in \mathcal{S}$. Each institution s holds onto applicants in $C_s\left(\mathcal{A}_s^1, q_s\right)$ and rejects the rest.

Step n$\geq$ 2. Each individual who was rejected in the previous step applies to the best institution that has not rejected him. Let $\mathcal{A}_s^n$ be the union of the set of individuals who were tentatively held by institution s at the end of Step $n-1$ and the set of new proposers of s in Step n. Each institution $s \in \mathcal{S}$ tentatively holds individuals in $C_s\left(\mathcal{A}_s^n, q_s\right)$ and rejects the rest.

The deferred acceptance algorithm terminates when there are no rejections. The outcome of the algorithm is the tentative assignments at that point. We denote the outcome by $\Phi(\widetilde{P}, \mathbf{q})$ to emphasize the dependence of the outcome on the vector of institutional reserve structures given by the profile $\mathbf{q}$. We denote $\Phi_i\left(\widetilde{P}, \mathbf{q}\right)$ be the assignment of individual i.

Baswana et al. (2018) report the new design for the joint seat allocation process for the technical universities in India that has been implemented since 2015.[8] Our focus is the sequential procedure introduced to deal with de-reservations, which is explained in detail in Chapter 6 of the technical report Baswana et al. (2019)[9].

According to this sequential process, the DA is first run with the initial capacities of reserve categories at each program. If there are unfilled seats that can be de-reserved in the resulting matching, then the capacities are updated by transferring the unfilled seats to the "*parent*" categories[10]. Then, the DA is *re-run* on all individuals with *updated* capacities of reserve categories at each institution. If there are no vacant seats that can be de-reserved in the resulting matching, then the process is terminated. This process is called *multi-run DA*.

We now formulate this procedure via the multiple iteration of the DA algorithm under choice rules with reserves.

Multi-run DA Algorithm

Let $\mathbf{q} = \mathbf{q}^1$ be the profile of initial distribution vector of positions over categories at institutions. Given a vector of reported preference relations of applicants $\widetilde{P} = \left(\widetilde{P}_i\right)_{i \in \mathcal{I}}$, a vector of reported reserve category membership $\mathbf{T} = (\mathbf{t}_i)_{i \in \mathcal{I}}$, and a profile of institutions' choice rules with reserves $\left(C_s^{Res}\right)_{s \in \mathcal{S}}$, the multi-run deferred acceptance algorithm runs as follows:

[8] Their design is a joint seat allocation process for IITs and non-IITs. The proposed mechanism is not a *direct* mechanism. Both IITs and non-IITs run their own DA algorithm for multiple rounds, in which applicant preferences are updated according to which program they accept. At the end of each round, candidates who accept a seat are provided three options: Float, Slide, or Freeze. Float means the applicant wants to be upgraded as high as possible on her preference list. Slide means the applicant wants to remain in the same institute but wants the most desirable program available. Freeze means the applicant wants to remain at the assigned program for the rest of the procedure. See Baswana et al. (2019) for algorithmic details. The authors also take other constraints into account, such as non-nested quotas, that we do not model for simplicity, as we mainly focus on the de-reservation policy.

[9] The report can be accessed at https://arxiv.org/pdf/1904.06698.pdf (last accessed on 02/24/2021).

[10] Open-category, for example, is a parent category for OBC.

Iteration 1. Run the DA with initial distributions of positions over categories $\mathbf{q}^1$. That is, for each institution $s \in \mathcal{S}$, use $C_s^{Res}\left(\cdot, q_s^1\right)$ to select applicants during the DA steps. Let r_s^1 be the number of vacant OBC seats. Update the number of open-category and OBC positions by transferring r_s^1 many positions from OBC to open-category. Let $\mathbf{q}^2$ be the profile of updated distributions of positions over categories.

Iteration n (n≥2). Run the DA with the updated distributions of reserved categories $\mathbf{q}^n$. That is, for each institution $s \in \mathcal{S}$, use $C_s^{Res}\left(\cdot, q_s^n\right)$ to select applicants during the DA steps. Let r_s^n be the number of vacant OBC seats. Update the number of open-category and OBC positions by transferring r_s^n many positions from OBC to open-category. Let $\mathbf{q}^{n+1}$ be the profile of updated distributions of positions over categories.

The algorithm terminates when there is no vacant position that can be de-reserved at any institution.

We denote the outcome of the multi-run DA by $\Phi\left(\widetilde{P}, \mathbf{q}^L\right)$, where L is the number of iterations needed, and $\mathbf{q}^L$ denotes the profile of updated distribution of positions at institutions in the last iteration. The outcome of individual $i \in \mathcal{I}$ is denoted by $\Phi_i\left(\widetilde{P}, \mathbf{q}^L\right)$.

To explain how distributions over reserve categories is updated during the multi-run DA algorithm, we now provide a simple example with a single institution.

Example 1. Suppose there are ten individuals with following category memberships and exam scores:

Individual	*Category*	*Merit Score*
i_1	*GC*	100
i_2	*GC*	99
i_3	*GC*	98
i_4	*GC*	97
i_5	*GC*	96
i_6	*OBC*	95
i_7	*SC*	94
i_8	*ST*	93
i_9	*GC*	92
i_{10}	*GC*	91

Consider institution s with ten positions. Suppose the initial distribution of positions over categories is

$$(OP, SC, ST, OBC) = (5, 1, 1, 3).$$

In the first iteration, the first five positions, i.e., open-category positions, are assigned to individuals i_1, i_2, i_3, i_4, and i_5. Individual i_6 is assigned to one of the three reserved positions for OBC. Two reserved OBC positions remain unfilled. The reserved position for SC is assigned to i_7. Similarly, the reserved position for ST is assigned to i_8. In total, eight individuals are assigned positions. Since two OBC positions remained vacant, the initial seat allocation is updated as $(7, 1, 1, 1)$.

In the second iteration, open-category positions are assigned to $\{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$. Individual i_8 is assigned to reserved ST position. Since i_9 and i_{10} are GC individuals, the reserved SC and OBC seats remain vacant. The vacant OBC seat is transferred to open-category. Hence, the new distribution over reserve categories becomes $(8, 1, 1, 0)$.

In the third iteration, open-category positions are assigned to $\{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\}$. Since i_9 and i_{10} are GC individuals, the reserved SC and ST seats remain unfilled. Therefore, even though there are individuals who are unassigned, two positions are wasted.

In this example, two of the three positions transferred from OBC to open-category are taken by SC and ST individuals. Since unfilled SC/ST positions cannot be transferred, this causes vacancies in reserve categories SC and ST. Therefore, the full benefit of de-reserving vacant OBC seats into open-category cannot be achieved. This example emphasizes the distributional consequences of the order at which categories are processed and the way de-reservations are implemented.

More importantly, when the DA is re-run to adjust the capacities of categories to handle de-reservations, it may **incentivize individuals to misreport their preferences**.

Proposition 1. *The multi-run DA mechanism is manipulable via preference misreporting.*

Proof. Suppose that there are two institutions a and b, each of which has two seats. Both schools reserve one seat for OBC candidates and consider the other seat as open-category. There are four applicants: i_1, i_2, i_3, and i_4. Suppose $\mathbf{t}_{i_1} = \mathbf{t}_{i_2} = GC$ and $\mathbf{t}_{i_3} = \mathbf{t}_{i_4} = OBC$. The merit scores of applicants are ranked from highest to lowest as $i_1 - i_2 - i_3 - i_4$. The true preferences of applicants are given below:

i_1	i_2	i_3	i_4
a	a	b	b
b	b	a	a
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

We first compute the outcome of the Multi-Run DA under the true preferences. In the first iteration of the deferred acceptance, applicants i_1 and i_2 are considered in institution a, while applicants i_3 and i_4 are considered in institution b. Since i_1 and i_2 are GC candidates, they are considered only for an open-category seat in institution a. i_1 is tentatively held for the open-category seat while i_2 is rejected. In institution b, applicant i_3 is tentatively held by the open-category seat and applicant i_4 is tentatively held by the OBC seat. Now, i_2 applies to b. Institution b holds i_2 for the open-category seat and i_3 for the OBC seat. Applicant i_4 is rejected from b in return. Next, i_4 applies a and is held by the OBC seat. The outcome is $\begin{pmatrix} a & b \\ \{i_1, i_4\} & \{i_2, i_3\} \end{pmatrix}$. The first iteration of the deferred acceptance is the final iteration and no de-reservation occurs. Note that i_2 is assigned her second choice institution.

Now, consider the following preferences, where i_2 misreports by stating a as the only acceptable alternative (i.e., she truncates).

i_1	i_2	i_3	i_4
a	a	b	b
b	$\emptyset$	a	a
$\emptyset$		$\emptyset$	$\emptyset$

In the first iteration of the deferred acceptance, applicants i_1 and i_2 are considered in institution a, while applicants i_3 and i_4 are considered in institution b. Since i_1 and i_2 are GC candidates, they are considered only for an open-category seat in institution a. i_1 is tentatively held for the open-category seat while i_2 is rejected. Since i_2 has no other institution to apply to, the deferred acceptance outcome of the first iteration is $\begin{pmatrix} a & b \\ \{i_1\} & \{i_3, i_4\} \end{pmatrix}$. Since there is a vacant OBC slot in institution a, it is de-reserved and the capacity of the open-category is set to 2 and deferred acceptance is re-run on all candidates. In the second iteration, both i_1 and i_2 are held by the open-category seats of a. Applicants i_3 and i_4 are held by b in open-category and OBC seats, respectively. Hence, the outcome of the second iteration is $\begin{pmatrix} a & b \\ \{i_1, i_2\} & \{i_3, i_4\} \end{pmatrix}$, where each applicant is assigned their top choices. Therefore, by misreporting, applicant i_2 receives a strictly better outcome. □

Moreover, the multi-run DA mechanism provides an advantage to individuals who can **strategize by not revealing their reserve category membership**.

Proposition 2. *The multi-run DA mechanism is manipulable via not reporting reserve category membership.*

Proof. Consider the same market in the proof of Proposition 1. The multi-run DA outcome is $\begin{pmatrix} a & b \\ \{i_1, i_4\} & \{i_2, i_3\} \end{pmatrix}$ when both i_3 and i_4 truthfully report their OBC membership under the given true preference profile. Now suppose that individual i_4 does not report her OBC membership, and, therefore, she is considered only for open-category positions.

In the first iteration of DA, individuals i_1 and i_2 apply to institution a, while applicants i_3 and i_4 apply to institution b in the first step. Since i_1 and i_2 are GC candidates, they are considered only for an open-category seat in institution a. i_1 is tentatively held for the open-category seat while i_2 is rejected. In institution b, both i_3 and i_4 are first considered for the open-category position. Since i_3 has a higher score, i_4 gets rejected. Note that since i_4 did not claim her OBC membership, she gets rejected from institution b. In the second step of the DA, i_2 applies to b and i_4 applies to a. At institution b, individual i_2 receives the open-category position by replacing i_3 and i_3 receives the reserved OBC slot. At institution a, i_1 keeps her open-category position and i_4 is rejected. Therefore, the first DA iteration results in the outcome $\begin{pmatrix} a & b \\ \{i_1\} & \{i_2, i_3\} \end{pmatrix}$. Since the OBC position in institution a remains unfilled, it is set as an open-category position for the second iteration of DA.

We now run the second DA iteration on all individuals. i_1 and i_2 apply to a, and they are both assigned to open-category positions since a has two open-category positions now. i_3 and i_4 apply to institution b. i_3 is assigned to the open-category position and i_4 gets rejected since she can be considered only for open-category positions and has a lower score than i_3. In the second step of the second iteration of DA, i_4 applies to her second choice, i.e., institution b. However, she gets rejected because her score is lower than both i_1 and i_2. Therefore, the second iteration DA outcome is $\begin{pmatrix} a & b \\ \{i_1, i_2\} & \{i_3\} \end{pmatrix}$. Since the OBC seat remains vacant in b, it is set to an open-category position so that b now has two open-category positions.

In the third iteration of DA, both a and b have two open-category positions. i_1 and i_2 apply to these and they are both assigned to open-category positions. i_3 and i_4 apply to b and they are both assigned to open-category positions. Therefore, the outcome of the third DA iteration is $\begin{pmatrix} a & b \\ \{i_1, i_2\} & \{i_3, i_4\} \end{pmatrix}$.

Note that when i_4 truthfully reveals her OBC category membership she was assigned to institution a, which is her second choice. However, when she does not reveal her OBC category membership she is assigned to her top choice, institution b. □

The purpose of the reservation policy is to protect the members of SC, ST, and OBC communities when they claim their privilege. This example, however, shows that it is possible for a reserved category member to get assigned to a better institution by not claiming her affirmative action privilege. This is in sharp contrast with the spirit of affirmative action. The main cause of this is the way de-reservation policy is implemented. Re-running the deferred acceptance algorithm to de-reserve unfilled slots from categories—which are allowed to be de-reserved to their "parent" categories—may create unnecessary rejection chains that in turn affect the distribution of positions over categories. This unnecessary change may incentivize individuals to misreport their caste membership to get a better assignment. We can, therefore, conclude that the de-reservation scheme in the multi-run DA mechanism may work against the core principle of the affirmative action policy.

Through Example 1 and Propositions 1 and 2, we reveal the unintended consequences of this particular de-reservation process. In the next section, we propose two different de-reservation schemes that will fix these shortcomings. Though we use the technical school admissions in India as our primary application, our proposals can be invoked in other resource allocation problems via reserve systems.

3 Backward and Forward Transfers Choice Rules

In this section, we formulate two classes of choice rules to implement both reserve and de-reservation policies. The deferred acceptance mechanism with respect to these choice rules will be

- strategy-proof, and
- incentive compatible for reserve category membership revelation.

Before introducing these choice rules, we first define the incentive compatibility for reserve category membership revelation *for choice rules*.

Definition 4. A choice rule C is **incentive compatible for reserve category membership revelation** if, for any given set of individuals $A \subseteq \mathcal{I}$ and any member of reserve category $r \in \mathcal{R}$ individual $i \in A$, if $i \notin C(A)$ when i reports her membership to category r, then $i \notin C(A)$ when i does not report her membership to r.

To put it differently, if individual i announces that $\mathbf{t}_i = GC$ and she is chosen from the set of individuals, then she must be chosen from the same set when she announces $\mathbf{t}_i = r \in \mathcal{R}$.

3.1 Backward Transfers Choice Rules

Given a set of applicants $A \subseteq \mathcal{I}$, a vector of reported reserve category membership $\mathbf{T} = (\mathbf{t_i})_{\mathbf{i}\in\mathcal{I}}$, and a vector of initial distribution of positions over reserve categories $q_s = q_s^1 = \left(q_s^1\right)$, the backward transfers choice rule C_s^{BT} selects applicants in multiple iterations as follows:

Iteration 1. Run the choice rule with reserves C_s^{Res} with q_s^1. That is, institution s selects applicants in $C_s^{Res}\left(A, q_s^1\right)$. Let τ^1 be the number of vacant OBC seats. Update the number of open-category and OBC positions by transferring τ^1 many positions from OBC to open-category. Let $q_s^2 = \left(q_s^2\right)_{s\in\mathcal{S}}$ be the vector of updated distributions of positions over reserve categories.

Iteration n (n≥2). Run the choice rule with reserves C_s^{Res} with the updated distribution of positions over reserve categories $q^n = \left(q_s^n\right)_{s\in\mathcal{S}}$. That is, institution s selects applicants $C_s^{Res}\left(A, q_s^n\right)$. Let τ^n be the number of vacant OBC seats. Update the number of open-category and OBC positions by transferring τ^n many positions from OBC to open-category. Let $q_s^{n+1} = \left(q_s^{n+1}\right)_{s\in\mathcal{S}}$ be the vector of updated distributions of positions over reserve categories.

The choice process terminates when there is no vacant OBC seat at any institution. The set of applicants who are selected in the last iteration—call it N—are the applicants who are selected by the backward transfers choice rule. That is,

$$C_s^{BT}(A, q_s) = C_s^{Res}(A, q^N).$$

Proposition 3. *Backward transfers choice rules are incentive compatible for reserve category membership revelation.*

Proof. See Appendix. □

We now present the DA algorithm with respect to backward transfers choice rules.

DA Algorithm under Backward Transfers Choice Rules

Let $\widetilde{P} = \left(\widetilde{P}_i\right)_{i \in \mathcal{I}}$ be the vector of *reported* preference relations and $\mathbf{T} = (\mathbf{t}_i)_{i \in \mathcal{I}}$ be the reported profile of reserve category membership of individuals. Given the backward transfers choice function of each institution $s \in \mathcal{S}$, C_s^{BT}—the outcome of the DA algorithm with respect to the backward transfers choice rules defined above is computed as follows:

Step 1. Each individual in $\mathcal{I}$ applies to his top choice institution. Let $\mathcal{A}_s^1$ be the set of individuals who applies to institution s, for each $s \in \mathcal{S}$. Each institution $s \in \mathcal{S}$ holds onto applicants in $C_s^{BT}\left(\mathcal{A}_s^1, q_s\right)$ and rejects the rest.

Step n≥ 2. Each individual who was rejected in the previous step applies to the best institution that has not rejected him. Let $\mathcal{A}_s^n$ be the union of the set of individuals who were tentatively held by institution s at the end of Step $n-1$ and the set of new proposers of s in Step n. Each institution $s \in \mathcal{S}$ tentatively holds individuals in $C_s^{BT}\left(\mathcal{A}_s^n, q_s\right)$ and rejects the rest.

The multi-run DA mechanism handles de-reservation by re-running the DA mechanism on all applicants to update the distribution of positions over categories by transferring unfilled OBC positions to the open-category, which is filled first according to the precedence sequence. In the DA mechanism with respect to backward transfers choice rules, de-reservations are handled by re-running the institutions' choice rules until there is no more vacancy to be de-reserved. Our first result shows that the DA mechanism with respect to these choice rules eliminates the possibility of manipulation via preference misreporting.

Theorem 1. *The DA mechanism with respect to backward transfer choice rules is strategy-proof.*

Proof. See Appendix. □

We prove Theorem 1 by showing that backward transfers choice rules satisfy classical substitutability and size monotonicity conditions. Substitutability requires that no two applicants i and j are complementary in the sense that the availability of j makes applicant i more desirable. Mathematically, a choice rule C is **substitutable** if for all $i, j \in \mathcal{I}$, and $A \subseteq \mathcal{I} \setminus \{i, j\}$,

$$i \notin C(A \cup \{i\}) \Longrightarrow i \notin C(A \cup \{i, j\}).$$

Size monotonicity requires that the number of chosen individuals weakly increases if the set of applicants expands. That is, a choice rule C is **size monotonic** if

$$A \subset A' \subseteq \mathcal{I} \Longrightarrow |C(A)| \leq |C(A')| .$$

Similar to the multi-run DA algorithm, under the DA algorithm with backward transfers vacant slots are transferred from the OBC category to the open-category that precedes the OBC category in the processing sequence. However, the two mechanisms are very different. According to the multi-run DA algorithm, the DA procedure is re-run with updated capacities of reserve categories on all individuals. According to the DA with respect to backward transfers choice rules, the procedure is run only once. De-reservations are handled by re-iterating the choice procedures of institutions in the course of the DA algorithm.

Unlike the multi-run DA mechanism, the DA mechanism with respect to backward transfers choice rules gives applicants incentive to report their reserve category membership truthfully.

Theorem 2. *The DA mechanism with respect to backward transfers choice rules is incentive compatible for reserve category membership revelation.*

Proof. See Appendix. □

There is a great benefit to re-running the choice rules rather than the DA algorithm to de-reserve vacant OBC positions. Consider the market in the proof of Proposition 1 with two institutions and four individuals. The Multi-run DA outcome was $\begin{pmatrix} a & b \\ \{i_1, i_4\} & \{i_2, i_3\} \end{pmatrix}$, where individuals i_1 and i_3 receive their top choices, while individuals i_2 and i_4 are assigned to their second choices under the true preferences. The outcome of the DA under backward transfers choice rules for the same market is $\begin{pmatrix} a & b \\ \{i_1, i_2\} & \{i_3, i_4\} \end{pmatrix}$, where all individual are assigned to their top choices. By re-iterating the choice rule rather than the DA algorithm, some unnecessary rejections chains are prevented. Our next result states that this observation can be generalized.

Theorem 3. *The DA mechanism with respect to backward transfers choice rules (weakly) Pareto dominates the multi-run DA mechanism at every problem P.*

Proof. See Appendix. □

Our proof for Theorem 3 is elegant. For the sake of brevity, we provide a sketch. We show that the outcome of the multi-run DA at a given preference profile $P = (P_i)_{i \in \mathcal{I}}$ is stable with respect to the backward transfers choice rules of institutions $\left(C_s^{BT}\right)_{s \in \mathcal{S}}$ at the same preference profile P given the profile of individuals' reserve category membership profile. Since backward transfers choice rules satisfy substitutes and size monotonicity properties, the DA outcome is *individual-optimal*. Given that multi-run DA and the DA with respect to backward transfers choice rules are different mechanisms, individual-optimality of the DA with respect to backward transfers choice rules implies that it Pareto dominates the multi-run DA.

Re-iterating the choice rule with reserves within the steps of a single-run DA rather than re-iterating the DA algorithm to update the distribution of positions not only gives applicants incentives to state their preferences truthfully, but also achieves a better outcomes for individuals at every problem. Therefore, using the DA with backward transfer choice rules is clearly a better choice between the two approaches.

We also have the following important corollary from the well-known Rural Hospital Theorem.

Corollary 1. *At every preference profile $P = (P_i)_{i \in \mathcal{I}}$, the number of individuals who are matched under the multi-run DA is the same as the number of individuals who are matched under the DA with respect to backward transfers choice rules.*

3.2 Forward Transfers Choice Rules

We now introduce forward transfers choice rules, according to which vacant positions are transferred from OBC to open-category by filling these extra open-category positions at the very end of the processing sequence. That is, the processing sequence becomes $Open \to (SC - ST - OBC) \to Open$. Forward transfer choice rules add a third stage to the choice rules with reserves. In the third stage, the surplus OBC positions are allocated as open-category positions. That is, given an initial distribution of positions $q_s = \left(q_s^{OP}, q_s^{SC}, q_s^{ST}, q_s^{OBC}\right)$, a set of applicants $A \subseteq \mathcal{I}$, and a vector of reported reserve category membership $\mathbf{T} = (\mathbf{t(i)})_{\mathbf{i} \in \mathcal{I}}$, the set of chosen applicants $C_s^{FT}(A, q_s)$, is computed as follows:

Stage 1. Only open-category seats are considered. Applicants are chosen one at a time following $\succ_s$ up to the capacity q_s^{OP}.

Stage 2. For each reserve category $t \in \mathcal{R}$, applicants are chosen one at a time following $\succ_s^t$ up to the capacity q_s^t. Let λ be the number of vacant OBC positions.

Stage 3. Applicants are chosen one at a time following $\succ_s$ up to the capacity λ.

We now give an example to show how the forward transfers choice rule is run.

Example 2. Let us reconsider the institution with ten positions in Example 1, where the initial distribution of positions over reserve categories is $(5,1,1,3)$. We will find the set of chosen individuals with respect to the forward transfers choice rule C_s^{FT} as follows: The first five positions, i.e., open-category positions are assigned to $\{i_1, i_2, i_3, i_4, i_5\}$. The reserved SC and ST positions are assigned to i_7 and i_8, respectively. One of the reserved OBC positions is assigned to i_6. Two reserved OBC positions remain vacant. These two positions are made open-category positions and individuals i_9 and i_{10} are assigned to them. All individuals are chosen under the choice rule C_s^{FT} while individuals i_9 and i_{10} were not chosen under C_s^{BT}.

Examples 1 and 2 reveal the crucial difference between the backward and forward transfers choice rules, which we discuss in detail in Section 4.

Proposition 4. *Forward transfers choice rules are incentive compatible for reserve category membership revelation.*

Proof. See Appendix. □

We now present the DA algorithm with respect to forward transfers choice rules.

DA Algorithm under Forward Transfers Choice Rules

Let $\widetilde{P} = \left(\widetilde{P}_i\right)_{i \in \mathcal{I}}$ be a vector of *reported* preference relations and $\mathbf{T} = (\mathbf{t}(\mathbf{i}))_{\mathbf{i} \in \mathcal{I}}$ be a vector of reported reserve category membership of individuals. Given the forward transfers choice function of each institution $s \in \mathcal{S}$, C_s^{FT}—the outcome of the DA algorithm with respect to the forward transfers choice rules defined above is computed as follows:

Step 1. Each individual in $\mathcal{I}$ applies to his top choice institution. Let $\mathcal{A}_s^1$ be the set of individuals who applies to institution s, for each $s \in \mathcal{S}$. Each institution s holds onto applicants in $C_s^{FT}\left(\mathcal{A}_s^1, q_s\right)$ and rejects the rest.

Step n≥ 2. Each individual who was rejected in the previous step applies to the best institution that has not rejected him. Let $\mathcal{A}_s^n$ be the union of the set of individuals who were tentatively held by institution s at the end of Step $n-1$ and the set of new proposers of s in Step n. Each institution $s \in \mathcal{S}$ tentatively holds individuals in $C_s^{FT}\left(\mathcal{A}_s^n, q_s\right)$ and rejects the rest.

The DA mechanism with respect to forward transfers choice rules gives applicants incentives to submit their true rankings over institutions.

Theorem 4. *The DA mechanism with respect to forward transfer choice rules are strategy-proof.*

Proof. See Appendix. □

Moreover, the DA mechanism with respect to forward transfers choice rules guarantees that reporting reserve category membership truthfully can never hurt.

Theorem 5. *The DA mechanism with respect to forward transfer choice rules is incentive compatible for reserve category membership revelation.*

Proof. See Appendix. □

4 Comparing Backward and Forward Transfers Choice Rules

To motivate our comparison, we start with the following example.

Example 3. Suppose there are eight individuals with the following reserved categories and exam scores:

Individual	*Category*	*Merit Score*
i_1	GC	100
i_2	GC	99
i_3	GC	98
i_4	SC	97
i_5	GC	96
i_6	OBC	95
i_7	ST	94
i_8	SC	93

Consider an institution s with seven positions, and the following initial distribution of positions over categories $(OP, SC, ST, OBC) = (3,1,1,2)$.

We first compute $C_s^{FT}(I, q_s)$. In the open-category i_1,i_2, and i_3 are selected, i.e., the three highest scoring candidates. For the reserved SC seat i_4 is selected. For the reserved ST seat i_7 is selected. For the reserved OBC seats, only i_6 is chosen and one OBC seat remains unfilled. Therefore, this vacant seat is made an open-category seat to be filled at the end. Among the unassigned individuals i_5 has the highest score, and she is selected for the extra open-category seat. Thus, we have

$$C_s^{FT}(I, q_s) = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$$

Note that the only individual who was not chosen is i_8, who has the lowest score.

We now compute $C_s^{BT}(I, q_s)$. In the *first* iteration, we compute $C_s^{Res}(I, q_s)$, i_1, i_2, and i_3 are selected in the open-category, i_4 is selected in the reserved SC category, i_7 is selected in the reserved ST category, and the only OBC individual i_6 is chosen in the reserved OBC category. One reserved OBC seat remains vacant. Therefore, it is made an open-category position by altering the vector of slot distribution across reserve categories. The new vector is $q_s^2 = (4,1,1,1)$.

In the *second* iteration, we compute $C_s^{Res}\left(I, q_2^s\right)$ as follows: i_1, i_2 , i_3, and i_4 are selected in the open-category, i_8 is selected in the reserved SC category, i_7 is selected in the reserved ST category, and i_6 is chosen in the reserved OBC category. Since there is no vacancy in the OBC category, we terminate the procedure. Thus, the set of chosen individuals is

$$C_s^{BT}(I, q_s) = \{i_1, i_2, i_3, i_4, i_6, i_7, i_8\}$$

Note that $C_s^{FT}(I, q_s) \setminus C_s^{BT}(I, q_s) = \{i_5\}$ and $C_s^{BT}(I, q_s) \setminus C_s^{FT}(I, q_s) = \{i_8\}$. So, i_5 is replaced with i_8 under $C_s^{BT}(I, q_s)$. The reason for this replacement is that the backward transferred OBC slot is taken by an SC individual i_4. In turn, the lowest scoring individual who belongs to SC is now chosen for the reserved SC category.

In this example, $C_s^{FT}(I, q_s)$ selects a better set of individuals than $C_s^{BT}(I, q_s)$ with respect to merit. In this section, we generalize this example and compare the outcomes of forward and backward transfers choice rules with the same initial distribution of positions over categories on the basis of merit.

Definition 5. A set of individuals I is *a better set of individuals on the basis of merit than* a set of individuals J with $| I | \geq | J |$ at institution s if there exists an injection $g : J \longrightarrow I$ such that

1. for all $j \in J$, $\kappa(g(j), s) \geq \kappa(j, s)$, and,
2. there exists $j \in J$ such that $\kappa(g(j), s) > \kappa(j, s)$.

We now introduce a criterion to compare two choice rules on the basis of merit.

Definition 6. A choice rule C *merit-based dominates* another choice rule $C^{'}$ if, for all sets of individuals $I \subseteq \mathcal{I}$, either $C(I) \supseteq C^{'}(I)$ or $C(I)$ is a better set on the basis of merit than $C^{'}(I)$.

Note that, according to Definition 6, if a choice rule C always chooses a super set of what choice rule $C^{'}$ chooses from the same given set of individuals, then C is considered more meritorious. This can be interpreted as "more is better" and is consistent with the main motivation of the recent admissions reform in India. Policymakers' primary goal was to reduce the number of wasted positions. Our next result compares backward and forward choice rules according to our merit-based domination criterion.

Theorem 6. *Forward transfer choice rule $C_s^{FT}(\cdot, q_s)$ merit-based dominates the backward transfer choice rule $C^{BT}(\cdot, q_s)$.*

Proof. See Appendix. □

Theorem 6 suggests that, if there is only one institution, using forward transfers choice rules is better than using the backward transfers choice rules because the former assigns not only a (weakly) higher number but also a more meritorious set of individuals. This comparison does not hold for the outcomes of the DA mechanisms with respect to backward and forward transfers choice rules, respectively. We illustrate this point below with an example.

Example 4. Consider two institutions $\mathcal{S} = \{a, b\}$. Institution a has four positions with the initial distribution vector over categories $(1, 1, 1, 1)$. Institution b has one position that is an open-category position. There are five individuals $\mathcal{I} = \{i_1, i_2, i_3, i_4, i_5\}$ with the following test score ordering at both institutions:

$$\kappa(i_1, s) > \kappa(i_2, s) > \kappa(i_3, s) > \kappa(i_4, s) > \kappa(i_5, s),$$

for both $s = a$ and $s = b$. The individuals' reserve category membership reports are as follows: $\mathbf{t}(i_1) = GC$, $\mathbf{t}(i_2) = SC$, $\mathbf{t}(i_3) = ST$, $\mathbf{t}(i_4) = ST$, and $\mathbf{t}(i_5) = ST$. The individuals' preferences are given below:

i_1	i_2	i_3	i_4	i_5
a	a	a	a	a
b	b	b	b	$\emptyset$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	

We first compute the outcome of the DA under backward transfers choice rules. In the first iteration of DA, all individuals apply to institution a in the first step. i_1 is chosen from open-category, i_2 is chosen from SC, i_3 is chosen from ST, and i_4 and i_5 are rejected because the only available position is the reserved OBC position. Individual i_4 applies to institution b in the second step, and is chosen for the open-category position. Individual i_5 remains unassigned. Since the reserved OBC position in a remains vacant, it is set as an open-category position and we move to the second iteration. The updated distribution vector of institution a becomes $(2,1,1,0)$.

In the second iteration of DA, all individuals apply to institution a in the first step. Individuals i_1 and i_2 are assigned to open-category positions. i_5 is assigned to a SC position, and i_3 is assigned to a ST position. i_4 gets rejected and applies to institution b in the second step and is chosen for the open-category position. Therefore, the outcome of the DA under backward transfers choice rules is $\begin{pmatrix} a & b \\ \{i_1, i_2, i_3, i_5\} & \{i_4\} \end{pmatrix}$.

We now compute the outcome of the DA algorithm under forward transfers choice rules. In the first step, all candidates apply to a. i_1 is chosen from open-category, i_2 is chosen from SC, and i_3 is chosen from ST. Since there is no OBC candidate, the reserved OBC seat remains unfilled and is set as an open-category position. Among the remaining individuals, i_4 is assigned to this position and i_5 gets rejected from a. Since there is no other acceptable institution for i_5, the DA algorithm terminates with the following outcome: $\begin{pmatrix} a & b \\ \{i_1, i_2, i_3, i_4\} & \emptyset \end{pmatrix}$.

Consider institution b. The number of individuals assigned to b in the DA algorithm under forward transfers choice rules, which is zero, is strictly less than the number of individuals assigned to it by the DA algorithm under backward transfers choice rules. Moreover, the set that b is assigned to, $\{i_4\}$, merit-based dominates the set b is assigned to under the forward transfers choice rule, i.e., $\emptyset$.

Our next result shows that there is no Pareto comparison between the outcomes of DA algorithms with respect to backward and forward choice rules, respectively.

Proposition 5. *The DA mechanism with forward transfer choice rules and the DA mechanism with backward transfer choice rules are Pareto incomparable.*

Proof. Consider an institution s with four positions, where the initial distribution over reserve categories is given by

$$\left(q_s^{OP}, q_s^{SC}, q_s^{ST}, q_s^{OBC}\right) = (1, 1, 1, 1).$$

Suppose there are five individuals $I = \{i_1, i_2, i_3, i_4, i_5\}$ to be considered. Individuals i_1 and i_3 are members of GC, individuals i_2 and i_5 are members of SC, and individual i_4 is a member of ST. The merit scores of individuals are given by

$$\kappa(i_1, s) > \kappa(i_2, s) > \kappa(i_3, s) > \kappa(i_4, s) > \kappa(i_5, s).$$

In the first iteration of the backwards transfers choice rule, i_1 is assigned to an open-category position, i_2 is assigned to a reserved SC position, and i_4 is assigned to a reserved ST position. The reserved OBC position remains vacant, and is set as an open-category seat for the second iteration. For the second iteration, the distribution over reserve categories becomes $(2, 1, 1, 0)$. Therefore, in the second iteration, i_1 and i_2 are assigned to open-category seats, i_4 is assigned to the reserved ST seat, and i_5 is assigned to the reserved SC seat. Therefore, the backward transfers choice rule selects the set $\{i_1, i_2, i_4, i_5\}$.

The forward capacity transfers choice rules assigns i_1 to an open-category position, i_2 to the reserved SC position, and i_4 to the reserved ST position. The reserved OBC position remains vacant, and therefore is set to an open-category seat. This new open-category position is assigned to i_3 since she has the highest merit score among all individuals who were not yet assigned. Thus, the forward transfers choice rule selects the set $\{i_1, i_2, i_3, i_4\}$.

Individual i_5 receives a better outcome under the DA with respect to the backward transfers choice rule, while individual i_3 receives a better outcome under the DA with respect to the forward transfers choice rule. Therefore, they are Pareto incomparable. □

5 Relation to the Literature

This paper contributes to the literature on resource allocation problems in India with affirmative action constraints that have been recently studied by Aygün and Turhan (2017, 2020, and 2021), Baswana et al. (2018 and 2019), and Sönmez and Yenmez (2020 and 2021). We have already discussed the differences between this work and that of Baswana et al. (2018, 2019). Sönmez and Yenmez (2020, 2021) assume away de-reservations altogether, while de-reservation policies are the main focus of this paper. Aygün and Turhan (2020 and 2021) use forward transfers choice rules. This paper compares different de-reservation schemes, including the forward transfers choice rules. To the best of our knowledge, our paper is the first to compare and analyze different de-reservation policies in detail.

This paper contributes to the literature on lexicographic choice rules in the context of allocating multiple identical objects under a capacity constraint. Lexicographic choice rules are also studied by Kominers and Sönmez (2016), Chambers and Yenmez (2017, 2018), Westkamp (2013), Aygün and Turhan (2020, 2021), Echenique and Yenmez (2015), Doğan (2017), and Doğan and Yıldız (2020), among others.

Our paper also contributes to the literature on controlled school choice and diversity considerations in matching markets that are also studied by, among others, Abdulkadiroğlu and Sönmez (2003), Biro et al. (2010), Ehlers (2010), Hafalir et al. (2013), Ehlers et al. (2014), Westkamp (2013), Echenique and Yenmez (2015), Kamada and Kojima (2015), Aygün and Bó (2021), Fragiadakis and Troyan (2017), Nguyen and Vohra (2019), Echenique et al. (2020), and Aziz et al. (2021).

Finally, our paper contributes to the market design literature, where economists study policy relevant real-world allocation problems in different contexts. Some examples of such allocation problems include refugee resettlement (Andersson 2017, Delacrétaz et al. 2020, and Jones and Teytelboym 2017), assignment of arrival slots (Schummer and Vohra 2013, and Schummer and Abizada 2017), course allocation (Sönmez and Ünver 2010, Budish 2011, and Budish and Cantillon 2012), and organ allocation and exchange (Roth et al. 2014, Ergin et al. 2017 and 2020), among many others.

6 Conclusion

This paper discusses unintended consequences of de-reservation policy implemented for admissions to technical universities in India to point out that how de-reservation policies are implemented is consequential. We introduce new de-reservation policies by embedding them into institutions' choice rules—backward and forward capacity transfers choice rules— to alleviate these consequences. We propose the DA mechanism with respect to these choice rules and show that it is strategy-proof and incentive compatible for reserve category membership revelation. We compare backward and forward transfers choice rules and show that the latter not only assigns a (weakly) higher number of individuals, but also a better set of individuals on the basis of merit. We believe that these choice rules can be implemented in many real-world allocation problems in which de-reservation is necessary.

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7 APPENDIX

Proof of Proposition 3. Suppose that individual i, who belongs to reserve category $r \in \mathcal{R}$, is not chosen by $C_s^{BT}(\cdot, q_s)$ when she reports her membership to category r, that is $\mathbf{t}_i = r$. We need to show that she is not chosen when she reports $\mathbf{t}_i^{'} = GC$. When i reports $\mathbf{t}_i = r$ and not chosen, that means i gets rejected for open-category positions in every iteration of C_s^{BT}. When i reports $\mathbf{t}_i^{'} = GC$, she gets rejected for open category positions in every iteration because, she cannot change the set of applicants who apply for open-category positions and the number of unfilled OBC seats that are transferred to open-category at the end of each iteration.

Before proving Theorem 1, we first state and prove a lemma that will be useful. This lemma also helps to understand the dynamics of the backward transfers choice rules.

Lemma 1. *Given a set of applicants $A \subseteq \mathcal{I}$ and a vector of initial distribution of positions over categories q_s of institution $s \in \mathcal{S}$, N is the last iteration of the backward transfers choice rule $C_s^{BT}(A, q_s)$ if, and only if, either one of the following holds:*

(1) $\mid \left(C_s^{OP}(A, (q_s^N)^{OP}) \setminus C_s^{OP}(A, (q_s^1)^{OP})\right) \setminus A^{OBC} \mid = \tau^1$*, where τ^1 is the number of unfilled OBC positions at the end of iteration 1, and $A^{OBC} = \{i \in A \mid \mathbf{t}(i) = OBC\}$.*

(2) $\left(q_s^N\right)^{OP} = \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC}$.

Condition (1) says that the number of non-OBC individuals who are chosen for open-category positions in the N^{th} iteration, but are not chosen for open-category positions in the first iteration of the multi-run DA, is exactly equal to τ^1, which is the number of vacant

OBC positions at the end of the first iteration of the multi-run DA. Condition (2) says that all OBC positions are made open-category positions at the beginning of iteration N. That is, all OBC positions that are made open-category positions at iteration $N-1$ are taken by OBC candidates. Hence, the remaining reserved OBC positions are now vacant, and made open category positions at the beginning of iteration N.

Proof of Lemma 1. ($\Leftarrow$) If we have $\left(q_s^N\right)^{OP} = \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC}$, then it means $\tau^n = 0$ by construction. That is, the number of vacant OBC seats at iteration N is 0. Hence, by definition, N is the final iteration.

Now suppose that $\mid \left(C_s^{OP}(A,(q_s^N)^{OP}) \setminus C_s^{OP}(A,(q_s^1)^{OP})\right) \setminus A^{OBC} \mid = \tau^1$. First, note that the following equality holds in iteration 1:

$$\left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC} = \tau^1 + \mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^1\right)^{OP}\right) \mid .$$

Since the total number of OBC and open-category positions in every iteration remains unchanged, we have the following equality holding at iteration N:

$$\left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC} = \left(q_s^N\right)^{OP} + \left(q_s^N\right)^{OBC} = \tau^N + \mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^N\right)^{OP}\right) \mid .$$

Since $\mid \left(C_s^{OP}(A,(q_s^N)^{OP}) \setminus C_s^{OP}(A,(q_s^1)^{OP})\right) \setminus A^{OBC} \mid = \tau^1$, we have

$$\mid C_s^{OP}\left(A, \left(q_s^N\right)^{OP}\right) \setminus \left(C_s^{OP}\left(Y, \left(q_s^1\right)^{OP}\right) \cup A^{OBC}\right) \mid = \tau^1,$$

which implies

$$\mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^N\right)^{OP}\right) \mid - \mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^1\right)^{OP}\right) \mid = \tau^1.$$

Therefore, we have

$$\mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^N\right)^{OP}\right) \mid = \tau^1 + \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC} - \tau^1$$

$$= \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC} = \left(q_s^N\right)^{OP} + \left(q_s^N\right)^{OBC},$$

which implies $\tau^N = 0$. Thus, N is the last iteration.

($\Longrightarrow$) Let N be the last iteration of $C_s^{BT}(A, q_s)$. Toward a contradiction, suppose that neither (1) nor (2) holds. That is, in the final step

$$\left(q_s^N\right)^{OP} < \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC} \Longrightarrow \left(q_s^N\right)^{OBC} > 0,$$

and

$$\mid \left(C_s^{OP}(A, (q_s^N)^{OP}) \setminus C_s^{OP}(A, (q_s^1)^{OP})\right) \setminus A^{OBC} \mid \neq \tau^1,$$

which implies

$$\mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^N\right)^{OP}\right) \mid - \mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^1\right)^{OP}\right) \mid \neq \tau^1.$$

Thus, we have

$$\mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^N\right)^{OP}\right) \mid \neq \tau^1 + \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC} - \tau^1 = \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC}$$

$$\neq \left(q_s^N\right)^{OP} + \left(q_s^N\right)^{OBC}.$$

This implies that

$$\mid A^{OBC} \setminus C_s^{OP}\left(A, \left(q_s^N\right)^{OP}\right) \mid + \mid C_s^{OP}\left(Y, \left(q_s^N\right)^{OP}\right) \mid \neq \left(q_s^N\right)^{OP} + \left(q_s^N\right)^{OBC}.$$

Since $C_s^{OP}(\cdot, \cdot)$ is a q-responsive choice function, we have two cases to consider:

Case 1: $\mid C_s^{OP}\left(A, (q_s^N)^{OP}\right) \mid < (q_s^N)^{OP}$. In this case, all individuals are accepted by $C_s^{OP}\left(A, (q_s^N)^{OP}\right)$. Hence,

$$A^{OBC} \setminus C_s^{OP}\left(A, \left(q_s^N\right)^{OP}\right) = \emptyset.$$

This implies $\tau^N = (q_s^N)^{OBC} > 0$. That means N is not the final iteration. This is a contradiction.

Case 2: $\mid C_s^{OP}\left(A,\left(q_s^N\right)^{OP}\right)\mid=\left(q_s^N\right)^{OP}$. In this case,

$$\mid A^{OBC}\setminus C_s^{OP}\left(A,\left(q_s^N\right)^{OP}\right)\mid\neq\left(q_s^N\right)^{OBC},$$

since $N-1$ is not the final iteration of $C_s^{BT}(A,q_s)$ by construction, i.e., $\tau^{N-1}>0$, we have

$$(i)\ C_s^{OP}\left(A,\left(q_s^{N-1}\right)^{OP}\right)\subset C_s^{OP}\left(A,\left(q_s^N\right)^{OP}\right)$$
$$(ii)\ \mid A^{OBC}\setminus C_s^{OP}\left(A,\left(q_s^{N-1}\right)^{OP}\right)\mid=\left(q_s^{N-1}\right)^{OBC}-\tau^{N-1}=\left(q_s^N\right)^{OBC}$$

(i) and (ii) imply

$$A^{OBC}\setminus C_s^{OP}\left(A,\left(q_s^N\right)^{OP}\right)\subset A^{OBC}\setminus C_s^{OP}\left(A,\left(q_s^N\right)^{OP}\right)$$

and

$$\mid A^{OBC}\setminus C_s^{OP}\left(A,\left(q_s^N\right)^{OP}\right)\mid\leq\mid A^{OBC}\setminus C_s^{OP}\left(A,\left(q_s^{N-1}\right)^{OP}\right)\mid=\left(q_s^N\right)^{OBC}.$$

Then, by $\mid A^{OBC}\setminus C_s^{OP}\left(A,\left(q_s^n\right)^{OP}\right)\mid\neq\left(q_s^N\right)^{OBC}$, we have

$$\mid A^{OBC}\setminus C_s^{OP}\left(A,\left(q_s^N\right)^{OP}\right)\mid<\left(q_s^N\right)^{OBC},$$

which implies that $\tau^N>0$. Hence, N is not the final iteration. This is a contradiction. This ends the proof of Lemma 1.

Proof of Theorem 1. We prove Theorem 1 by showing that backward transfers choice rules satisfy both substitutability and size monotonicity.

Substitutability. Consider $i,j\in\mathcal{I}$ and $A\subset\mathcal{I}\setminus\{i,j\}$ such that $i\notin C_s^{BT}(A\cup\{i\})$. We need to show that $i\notin C_s^{BT}(A\cup\{i,j\})$.

Let τ^k and $\widetilde{\tau}^k$ denote the number of vacant OBC positions at the end of iteration k under $C_s^{BT}(A\cup\{i\},q_s)$ and $C_s^{BT}(A\cup\{i,j\},q_s)$, respectively. Let N and $\widetilde{N}$ be the last steps of $C_s^{BT}(A\cup\{i\},q_s)$ and $C_s^{BT}(A\cup\{i,j\},q_s)$, respectively. Note that, by Lemma 1, $\widetilde{N}\leq N$ and $\left(q_s^N\right)^{OP}\geq\left(q_s^{\widetilde{N}}\right)^{OP}$, where $\left(q_s^N\right)^{OP}$ and $\left(q_s^{\widetilde{N}}\right)^{OP}$ are the capacities of open-category at the last steps of $C_s^{BT}(A\cup\{i\},q_s)$ and $C_s^{BT}(A\cup\{i,j\},q_s)$, respectively. Let $A_t\subseteq A\cup\{i\}$ denotes the set of individuals who belong to reserve category $t\in\mathcal{R}=\{SC,ST,OBC\}$. For each $t \in \mathcal{R}$, define $A_t' = A_t \setminus C_s^{OP}\left(A \cup \{i\}, (q_s^N)^{OP}\right)$.

If $i \notin C_s^{BT}(A \cup \{i\}, q_s)$, then we know that i is not in the top $\left(q_s^N\right)^{OP}$ in the set $A \cup \{i\}$. This implies that i is not in top $\left(q_s^{\widetilde{N}}\right)^{OP}$ in the set $A \cup \{i, j\}$. So, i cannot be chosen for an open-category position from $A \cup \{i, j\}$.

We now show that i cannot be chosen for a reserve category $\mathbf{t}(i) \in \mathcal{R}$ seat. First, suppose that $\mathbf{t}(i) = OBC$. Since i was not chosen for an OBC position from the set $A \cup \{i\}$, and when j is added to the set $A \cup \{i\}$, i cannot be chosen for an OBC position because adding j (weakly) increases the competition for OBC positions.

Now, suppose that $\mathbf{t}(i) \in \{SC, ST\}$. The capacities of reserved SC and ST categories are the same for the choice processes starting with $A \cup \{i\}$ and $A \cup \{i, j\}$ in every iteration of the C_s^{BT}. Moreover, we have

$$A \cup \{i, j\} \setminus C_s^{OP}\left(A \cup \{i, j\}, \left(q_s^{\widetilde{N}}\right)^{\mathbf{t}(i)}\right) \supseteq A \cup \{i\} \setminus C_s^{OP}\left(A \cup \{i\}, \left(q_s^{N}\right)^{OP}\right),$$

for both $t = SC$ and $t = ST$. That is, the competition for the SC and ST positions will be (weakly) higher in the choice process starting with $A \cup \{i, j\}$ than the choice process starting with $A \cup \{i\}$. Since i was not chosen for reserved $\mathbf{t}(i)$ position from $A \cup \{i\} \setminus C_s^{OP}\left(A \cup \{i\}, (q_s^N)^{OP}\right)$, we can conclude that i will not be chosen for reserved $\mathbf{t}(i)$ position from $A \cup \{i, j\} \setminus C_s^{OP}\left(A \cup \{i, j\}, \left(q_s^{\widetilde{N}}\right)^{\mathbf{t}(i)}\right)$. Therefore, i cannot be chosen for reserved $\mathbf{t}(i) \in \mathcal{R}$ positions. This ends our proof of substitutability.

Size monotonicity. Consider $i \in \mathcal{I}$ and $A \subseteq \mathcal{I} \setminus \{i\}$. We need to show that $\mid C_s^{BT}(A) \mid \leq \mid C_s^{BT}(A \cup \{i\}) \mid$. We consider following two cases:

Case 1. $\mid A \mid \leq q_s^{OP} + q_s^{OBC}$**.** In this case, all individuals in A will be chosen. When i is added to the set A, the number of chosen individuals increases by one and becomes $\mid A \mid +1$, if $\mid A \mid < q_s^{OP} + q_s^{OBC}$. When $\mid A \mid \leq q_s^{OP} + q_s^{OBC}$, since the number of chosen individuals is $\mid A \mid$, adding i to the set A does not change the number of chosen individuals.

Case 2. $\mid A \mid > q_s^{OP} + q_s^{OBC}$**.** The backward transfers choice rule C_s^{BT} selects $min\left\{\mid A \mid, \overline{q}_s\right\}$ individuals, unless either $\mid A_{SC}' \mid < q_s^{SC}$ or $\mid A_{ST}' \mid < q_s^{ST}$, where $A_t' = A_t \setminus C_s^{OP}\left(A, (q_s^N)^{OP}\right)$ for $t \in \{SC, ST\}$. Note that N represents the last iteration of the backward transfers choice rule. In other words, the choice rule C_s^{BT} behaves as a q-responsive choice function if the number of remaining SC and ST individuals after open-category positions are filled are at least as many as the number of reserved SC and ST positions, respectively. Therefore, when both $\mid A'_{SC} \mid \geq q_s^{SC}$ or $\mid A'_{ST} \mid \geq q_s^{ST}$, the number of chosen individuals will be $\overline{q}_s$ and adding individual i to the set A does not change the number of chosen individuals. When either $\mid A'_{SC} \mid < q_s^{SC}$ or $\mid A'_{ST} \mid < q_s^{ST}$, the number of chosen individuals from $A \cup \{i\}$ either stays the same or increases by one.

Substitutability and size monotonicity of backward transfers choice rules imply strategy-proofness of the DA with respect to backward transfers choice rules, following Hatfield and Milgrom (2005). This ends our proof.

Proof of Theorem 2. We adapt the following definition from Definition 8 of Afacan (2017): A choice rule C'_s is an *improvement* over a choice rule C_s for individual i if, for any set of individuals A (i) if $i \in C_s(A)$, then $i \in C'_s(A)$, and (ii) if $i \notin C_s(A) \cup C'_s(A)$, then $C_s(A) = C'_s(A)$. Consider a reserve category $r \in \mathcal{R}$ member i. Let C_s^{BT} be the backward transfers choice rule individual i faces when she does *not* report her category r membership, and $\widetilde{C}_s^{BT}$ be the backward transfers choice rule individual i faces when she *reports* her membership to r. Our Proposition 3 states that $\widetilde{C}_s^{BT}$ is an improvement over C_s^{BT} for individual i according to the given definition of improvement.

Mechanism ψ *respects improvements* if for any problem (P, C) and C' such that C' is an improvement over C for individual i, $\psi(P, C') R_i \psi(P, C)$.

Theorem 2 of Afacan (2017) states that the generalized DA respects improvements if choice rules of institutions satisfy the *unilateral substitutes* of Hatfield and Kojima (2010), the *irrelevance of rejected contracts* of Aygün and Sönmez (2013), and the *size monotonicity* of Hatfield and Milgrom (2015). As we show in Theorem 1, backward transfers choice rules satisfy substitutability. In our setting without contracts, substitutability and unilateral substitutability are equivalent. Moreover, backward transfers choice rules satisfy size monotonicity, which—in conjunction with substitutability—implies the irrelevance of rejected contracts condition. Therefore, Afacan (2017)'s Theorem 2 hold in our setting with backward transfers choice rules. Then, we can conclude that when individuals report their reserved category membership, they can never be hurt under the DA mechanism with respect to backward transfers choice rules.

Proof of Theorem 3. As it was shown in the proof of Theorem 1, backward transfers choice rules are substitutable and size monotonic. Therefore, by Theorem 4 of Hatfield and Milgrom (2005), the generalized DA outcome is the *individual-optimal* stable outcome, where stability is defined with respect to a profile of backward transfers choice rules $\left(C_s^{BT}\right)_{s\in\mathcal{S}}$. That is, each applicant weakly prefers the outcome of the generalized DA to her assignment in any other stable matching.

Let v be the outcome of the multi-run DA at preference profile $P = (P_i)_{i\in\mathcal{I}}$. That is, $v = \Phi\left(P, q^L\right)$, where L denotes the last iteration of the DA in multi-run DA algorithm. We will show that v is stable with respect to the profile of backward transfers choice rules $\left(C_s^{BT}\right)_{s\in\mathcal{S}}$.

Individual Rationality for Individuals. Since the preference profile in the multi-run DA and the DA with respect to backward transfers choice rules are the same, for every individual $i \in \mathcal{I}$, $v(i) R_i \emptyset$.

Individual Rationality for Institutions. We need to show that the outcome of the multi-run DA at preference profile P is individually rational for institution $s \in \mathcal{S}$ with respect to its backward transfers choice rule C_s^{BT}, for all institutions $s \in \mathcal{S}$. That is, $C_s^{BT}(v(s)) = v(s)$ for all institutions $s \in \mathcal{S}$, where $v(s)$ denotes the set of applicants who are matched to institution s under the multi-run DA.

We will first prove a lemma that will be the key to prove individual rationality for institutions. We introduce the necessary notation first. Consider a set of applicants $A \subseteq \mathcal{I}$. Let A^{SC}, A^{ST}, and A^{OBC} be sets of individuals who belong to SC, ST, and OBC, respectively, in the set A. In the backward transfers choice rules, let $q_s = q_s^1$ be the initial vector of capacities of categories, N be the last iteration, and q_s^N denote the vector of capacities of categories at institution $s \in \mathcal{S}$ in the last iteration of C_s^{BT}. By definition, $C_s^{BT}(A, q_s) = C^{Res}\left(A, q_s^N\right)$. We denote by $C_s^{OP}\left(A, \left(q_s^n\right)^{OP}\right)$ the set of individuals chosen from the open category given a set of applicants A , and capacity $\left(q_s^n\right)^{OP}$ of the open-category at iteration n of the backward transfers choice rule. This choice rule selects applicants following the priority ordering $>_s$ of institution s up to the capacity $\left(q_s^n\right)^{OP}$.

By Lemma 1, $C_s^{BT}(A, q_s)$ is finalized as soon as either one of the conditions in Lemma 1 is satisfied. In the last iteration, call it L, of the multi-run DA we have either

$$| C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right) \cup A^{OBC} |= \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC}$$

or

$$\left(q_s^L\right)^{OP} = \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC}.$$

Then, we have $\left(q_s^L\right)^{OP} \geq \left(q_s^n\right)^{OP}$, which implies

$$C_s^{OP}\left(A, \left(q_s^n\right)^{OP}\right) \subseteq C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right),$$

which, in turn, implies

$$A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^n\right)^{OP}\right) \subseteq A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right).$$

Hence, we have

$$\mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^n\right)^{OP}\right) \mid \leq \mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right) \mid.$$

By Lemma 1 and the fact that L satisfies either $\mid C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right) \cup A^{OBC} \mid = \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC}$ or $\left(q_s^L\right)^{OP} = \left(q_s^1\right)^{OP} + \left(q_s^1\right)^{OBC}$ in multi-run DA, we conclude

$$\mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^n\right)^{OP}\right) \mid = \mid A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right) \mid.$$

Moreover, since $C_s^{OP}(\cdot,\cdot)$ is q-responsive, we have

$$A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^n\right)^{OP}\right) = A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right),$$

which implies

$$A^{OBC} \cup C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right) \subseteq C_s^{BT}(A, q_s).$$

Moreover, by the construction of multi-run DA we have

$$\mid A^{SC} \setminus C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right) \mid \leq \left(q_s^1\right)^{SC}$$
$$\mid A^{ST} \setminus C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right) \mid \leq \left(q_s^1\right)^{ST}$$

because there are no de-reservations from categories SC and ST, and hence, capacities of these categories remain unchanged in the course of multi-run DA. Thus, we have

$$A^{SC} \setminus C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right) \subseteq C_s^{BT}\left(A, q_s\right)$$
$$A^{ST} \setminus C_s^{OP}\left(A, \left(q_s^L\right)^{OP}\right) \subseteq C_s^{BT}\left(A, q_s\right)$$

which completes our proof. Hence, the outcome of the multi-run DA at preference profile P, i.e., the matching ν, is individually rational for every institution with respect to their backward transfers choice rules.

No Blocking. Toward a contradiction, suppose that (i,s) is a blocking pair. That is, $sP_i\nu(i)$ and $i \in C_s^{BT}(\nu(s) \cup \{i\})$. Let A_s^{SC}, A_s^{ST}, and A_s^{OBC} denote the set of applicants in $\nu(s)$ that are members of SC, ST, and OBC, respectively. There are three cases to consider.

(i) $OBC = \mathbf{t}(i)$. Since $sP_i\nu(i)$ and OBC applicants get weakly better of in multi-run DA, we can conclude that individual i applied to s and get rejected by s in every iteration of the multi-run DA. Since i was never chosen, we know that the number of chosen OBC members is at least as high as the initial capacity of the OBC category in every iteration. Therefore, $\mid A_s^{OBC} \mid \geq q_s^{OBC}$. Moreover, individual i is not in top q_s^{OP} candidates in the set $\nu(s)$ and every applicant in A_s^{OBC} has higher merit score than i. Thus, i cannot be chosen from $\nu(s) \cup \{i\}$ in the backward transfers choice rule C_s^{BT}. This contradicts with (i,s) being a blocking pair.

(ii) $SC = \mathbf{t}(i)$ or $ST = \mathbf{t}(i)$. Let us consider $SC = \mathbf{t}(i)$. Since $sP_i\nu(i)$ and SC applicants get weakly better of in multi-run DA, we can conclude that individual i applied to s and get rejected by s in every step of the multi-run DA. Since i is not chosen by s, all candidates in $A_s^{GC} \cup A_s^{SC}$ have higher scores than i. Moreover, there is no unfilled seat at reserved SC category. Since adding SC candidates who have lower scores than candidates in $A_s^{GC} \cup A_s^{SC}$ to $\nu(s)$ cannot change the capacity vector of the final iteration of C_s^{BT}, i cannot be chosen from $\nu(s) \cup \{i\}$. This is a contradiction. The case where $ST = \mathbf{t}(i)$ is proved similarly.

(iii) $\{GC\} = \mathbf{t}(i)$. Since $sP_i\nu(i)$ and GC applicants get weakly better of in multi-run DA, individual i applied to s and get rejected by s in every iteration of the multi-run DA. Since i was never chosen, we know that i is not in top $\left(q^N\right)_s^{OP}$, i.e., number of open-category seats in the final iteration of the multi-run DA. Therefore, i cannot be chosen from $\nu(s) \cup \{i\}$. This is a contradiction. Therefore, there is no blocking pair.

We have shown that the outcome of the multi-run DA at problem P, i.e., matching ν, is stable with respect to backward transfers choice rules C^{BT}. Therefore, the applicantoptimal stable (with respect to backward transfers choice rules C^{BT}) matching μ Pareto dominates matching ν.

Proof of Proposition 4. Suppose that individual i, who is a reserve category $r \in \mathcal{R}$ member, is not chosen by $C_s^{FT}(\cdot, q_s)$ when she reports her membership to category $r \in \mathcal{R}$, that is $\mathbf{t}_i = r$. We need to show that she is not chosen when she reports $\mathbf{t}_i' = GC$. When i reports $\mathbf{t}_i = r$ and not chosen, that means i gets rejected for open-category positions under C_s^{FT}. When i reports $\mathbf{t}_i' = GC$, she gets rejected for open category positions, because she cannot change the set of applicants who apply for open-category positions and the number of unfilled OBC seats that are transferred to open-category to be filled at the very end.

Proof of Theorem 4. In forward transfer choice rules, vacant OBC seats are transferred to open-category that succeed the reserve category OBC. We can write the capacity of the extra open-category positions that are filled at the end as

$$q^{EOP}(r_{OP}, r_{SC}, r_{ST}, r_{OBC}) = r_{OBC},$$

where r_{OP}, r_{SC}, r_{ST}, and r_{OBC} denote the number of unfilled slots in categories open-category, SC, ST, and OBC, respectively. This choice protocol straightforwardly satisfies the monotonicity and non-excessive reduction properties of Westkamp (2013), which imply the substitutability and size monotonicity of forward transfers choice rules. By Theorem 2 of Westkamp (2013), the DA with respect to forward transfers choice rules is strategy-proof for individuals.

Proof of Theorem 5. Let C_s^{FT} be the forward transfers choice rule individual i faces when she does not report her category r membership, and $\widetilde{C}_s^{FT}$ be the forward transfers choice rule individual i faces when she reports her membership to r. Our Proposition 4 states that $\widetilde{C}_s^{FT}$ is an improvement over C_s^{FT} for individual i according to the improvement notion defined in the proof of Theorem 2. Since forward transfers choice rules satisfy substitutability and size monotonicity, and hence the irrelevance of rejected contracts condition, we can invoke Theorem 2 of Afacan (2017). Therefore, when individuals report their reserved category membership, they can never be hurt under the DA mechanism with respect to forward transfers choice rules.

Proof of Theorem 6. Consider an institution $s \in \mathcal{S}$ and a set of individuals $A \subseteq \mathcal{I}$. Let $q_s = \left(q_s^{OP}, q_s^{SC}, q_s^{ST}, q_s^{OBC}\right)$ be the vector of initial distribution of capacities over categories. Let $\left((q_s^N)^{OP}, q_s^{SC}, q_s^{ST}, (q_s^N)^{OBC}\right)$ and $\left(q_s^{OP}, q_s^{SC}, q_s^{ST}, q_s^{OBC}, \lambda\right)$ be the final capacity vectors of categories under the backward and forward transfers choice rules, C_s^{BT} and C_s^{FT}, respectively. Note that N denotes the final iteration of C_s^{BT} and λ denotes the number of unfilled OBC seats that are converted into open-category positions under C_s^{FT}.

Let $A_t \subseteq A$ denotes the set of individuals who belong to reserve category $t \in \mathcal{R} = \{SC, ST, OBC\}$. For each $t \in \mathcal{R}$, define

$$A_t' = A_t \setminus C_s^{OP}\left(A, q_s^{OP}\right)$$
$$\widetilde{A}_t = A_t \setminus C_s^{OP}\left(A, (q_s^N)^{OP}\right)$$

First note that by definition of the backward transfers choice rule C_s^{BT}, we have

$$C_s^{BT}(A, q_s) = C_s^{OP}\left(A, (q_s^N)^{OP}\right) \cup C_s^{SC}\left(\widetilde{A}_{SC}, q_s^{SC}\right) \cup C_s^{ST}\left(\widetilde{A}_{ST}, q_s^{ST}\right) \cup C_s^{OBC}\left(\widetilde{A}_{OBC}, (q_s^N)^{OBC}\right) \quad (1)$$

Similarly, by the definition of the forward transfers choice rule C_s^{BT}, we have

$$C_s^{FT}(A, q_s) = C_s^{OP}\left(A, q_s^{OP}\right) \cup C_s^{SC}\left(A_{SC}', q_s^{SC}\right) \cup C_s^{ST}\left(A_{ST}', q_s^{ST}\right) \cup C_s^{OBC}\left(A_{OBC}', q_s^{OBC}\right) \cup C_s^{OP}(R, \tau_1) \quad (2)$$

where R is the set of remaining individuals, i.e., $R = A \setminus C_s^{Res}(A, q_s)$.

Since we have $(q_s^N)^{OP} \geq q_s^{OP}$ and each category chooses individuals following the merit scores up to the capacity, i.e., they are all q-responsive choice functions, we have

$$C_s^{OP}\left(A, q_s^{OP}\right) \subseteq C_s^{OP}\left(A, (q_s^N)^{OP}\right).$$

For each $t \in \{SC, ST\}$, since $C_s^t\left(A_t', q_s^t\right)$ is the top q_s^t candidates, we have the following: for each individual $i \in C_s^{OP}\left(A_t', q_s^t\right)$, we have

$$i \in C_s^{OP}\left(A, (q_s^N)^{OP}\right) \cup C_s^t\left(\widetilde{A}_t, q_s^t\right),$$

which implies $C_s^t\left(A_t', q_s^t\right) \subseteq C_s^{BT}(A, q_s)$. That is, every SC and ST individual who are chosen from the reserved SC and ST categories under the forward transfers choice rule are also chosen under the backward transfers choice rule.

Let

$$m_t = \left| \left(C_s^{OP}\left(A, (q_s^N)^{OP}\right) \setminus C_s^{OP}\left(A, q_s^{OP}\right)\right) \cap A_t \right|,$$

for $t \in \{OP, SC, ST\}$. By Lemma 1, we have either

(i) $m_{OP} + m_{SC} + m_{ST} = \tau_1$, or

(ii) $m_{OP} + m_{SC} + m_{ST} < \tau_1$ and $C_s^{BT}(A, q_s) = A$.

In the case of (ii), we have $|A| < q_s^{OP} + q_s^{OBC}$. This implies $C_s^{FT}(A, q_s) = A$. For the case (i), we have

$$| C_s^{OP}\left(A, (q_s^N)^{OP}\right) | + | C_s^{OBC}\left(\widetilde{A}_{OBC}, (q_s^N)^{OBC}\right) | = q_s^{OP} + q_s^{OBC} \qquad (3)$$

which implies

$$| C_s^{OP}\left(A, q_s^{OP}\right) | + | C_s^{OBC}\left(A'_{OBC}, (q_s^N)^{OBC}\right) | = q_s^{OP} + q_s^{OBC} - \tau_1 \qquad (4)$$

We also have the following inequalities

$$0 < | C_s^{SC}\left(A'_{SC}, q_s^{SC}\right) | - | C_s^{SC}\left(\widetilde{A}_{SC}, q_s^{SC}\right) | < m_{SC} \qquad (5)$$

$$0 < | C_s^{ST}\left(A'_{ST}, q_s^{ST}\right) | - | C_s^{ST}\left(\widetilde{A}_{ST}, q_s^{ST}\right) | < m_{ST} \qquad (6)$$

Given the equalities (1) and (2), summing over equalities (3) and (4) and inequalities (5) and (6), we have the following inequality:

$$-\tau_1 < | C_s^{FT}(A, q_s) | - | C_s^{OP}(R, \tau_1) | + | C_s^{BT}(A, q_s) | < m_{SC} + m_{ST} - \tau_1 \qquad (7)$$

There are two cases to consider given inequality (7):

Case 1. $| C_s^{OP}(R, \tau_1) | = | R |$. In this case, we have $C_s^{FT}(A, q_s) = A \supseteq C_s^{BT}(A, q_s)$. This implies $| C_s^{FT}(A, q_s) | \geq = | C_s^{BT}(A, q_s) |$.

Case 2. $| C_s^{OP}(R, \tau_1) | = \tau_1$. In this case, from inequality (7), we have

$$0 < | C_s^{FT}(A, q_s) | - | C_s^{BT}(A, q_s) | < m_{SC} + m_{ST},$$

which also implies $| C_s^{FT}(A, q_s) | \geq = | C_s^{BT}(A, q_s) |$.

Therefore, for any given set $A \subseteq \mathcal{I}$, the forward transfers choice rule selects at least as many individuals backward transfers choice rule selects, i.e.,

$$\mid C_s^{FT}(A, q_s) \mid \geq \mid C_s^{BT}(A, q_s) \mid .$$

We now show that $C_s^{FT}(A, q_s)$ merit-based dominates $C_s^{BT}(A, q_s)$. First, note that we can write the set of selected individuals under the forward transfers choice rule $C_s^{FT}(A, q_s)$ as

$$C_s^{Res}(A, q_s) \cup C_s^{OP}\left(R, \tau^1\right),$$

where $R = A \setminus C_s^{Res}(A, q_s)$ and τ^1 is the number of unfilled OBC positions that are made open-category positions (and, filled at the very end of the processing sequence). We have already shown that if $i \in C_s^{Res}(A, q_s)$, then $i \in C_s^{BT}(A, q_s)$. That is, any individual who are chosen either in Stage 1 or Stage 2 of the forward transfers choice rule is chosen by the backward transfer choice rule. We will now construct an injection $g : C_s^{BT}(A, q_s) \rightarrow C_s^{FT}(A, q_s)$. For all individuals in $C_s^{Res}(A, q_s)$, which is a subset of both $C_s^{FT}(A, q_s)$ and $C_s^{BT}(A, q_s)$, we set $g(i) = i$.

Since we have $\mid C_s^{FT}(A, q_s) \mid \geq \mid C_s^{BT}(A, q_s) \mid$, we also have

$$\mid C_s^{FT}(A, q_s) \setminus C_s^{Res}(A, q_s) \mid \geq \mid C_s^{BT}(A, q_s) \setminus C_s^{Res}(A, q_s) \mid .$$

We call $C_s^{FT}(A, q_s) \setminus C_s^{Res}(A, q_s) \mid$ as $C_s^{OP}\left(R, \tau^1\right)$, which is a q-responsive choice function. That is, among the remaining individuals either all of them or the top τ^1 of them will be chosen following the merit scores.

Given the set of individuals $C_s^{BT}(A, q_s) \setminus C_s^{Res}(A, q_s)$ and $C_s^{FT}(A, q_s) \setminus C_s^{Res}(A, q_s)$, we map the top-scoring individual in $C_s^{BT}(A, q_s) \setminus C_s^{Res}(A, q_s)$, call her j_1, to the top-scoring individual in $C_s^{FT}(A, q_s) \setminus C_s^{Res}(A, q_s)$, call her $\widetilde{j_1}$. That is, $g(j_1) = \widetilde{j_1}$. We map the second top-scoring individual in $C_s^{BT}(A, q_s) \setminus C_s^{Res}(A, q_s)$, call her j_2, to the second top-scoring individual in $C_s^{FT}(A, q_s) \setminus C_s^{Res}(A, q_s)$, call her $\widetilde{j_2}$. That is, $g(j_2) = \widetilde{j_2}$. We process in the same fashion, until we exhaust all individuals in the set $C_s^{BT}(A, q_s) \setminus C_s^{Res}(A, q_s)$, in K steps. If $\{j_1, ..., j_K\} = \left\{\widetilde{j_1}, ..., \widetilde{j_K}\right\}$, then we will have

$$C_s^{FT}(A, q_s) \setminus C_s^{Res}(A, q_s) \supseteq C_s^{BT}(A, q_s) \setminus C_s^{Res}(A, q_s),$$

which implies

$$C_s^{FT}(A, q_s) \supseteq C_s^{BT}(A, q_s).$$

In this case, we are done according to the merit-based domination definition. Otherwise, i.e., $\{j_1, ..., j_K\} \neq \left\{\widetilde{j}_1, ..., \widetilde{j}_K\right\}$, then there must exist an individual in $C_s^{FT}(A, q_s) \setminus C_s^{Res}(A, q_s)$, call her $g(j^{'})$, who has a strictly higher score than the individual $j^{'}$ in $C_s^{BT}(A, q_s) \setminus C_s^{Res}(A, q_s)$ because $C_s^{OP}\left(R, \tau^1\right)$, which is a q-responsive choice function. Therefore, in this case, $C_s^{FT}(A, q_s) \mid$ is a better set of individuals on the basis of merit than $C_s^{BT}(A, q_s)$. This ends our proof.