Target the vulnerable?
An analysis of rapid rehousing prioritization

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Abstract

We model the problem facing a policymaker who must allocate rapid rehousing support to people experiencing homelessness and wishes to minimize the steady-state size of the homeless population. Typically, support is given to the most vulnerable applicants, or to applicants most likely to remain housed. We show that these approaches may result in a homeless population that is arbitrarily larger than what could be achieved by an optimal policy, and propose an alternative priority queue that is approximately optimal.

We then study a family of policies where the policymaker does not differentiate between agents based on their characteristics. Within this family, FIFO queues best target the most vulnerable. If the most vulnerable households benefit most from housing assistance, then a FIFO queue minimizes the expected unhoused population. Conversely, a LIFO queue is optimal if the least vulnerable households benefit most from housing assistance.

1 Introduction

1.1 Background on Rapid Rehousing

Rapid Rehousing. Rapid rehousing (RRH) programs offer housing assistance to households experiencing homelessness. These households are assigned to a case manager, who helps them find and pay for rental housing. Unlike other rental assistance programs such as Section 8 Housing Choice Vouchers, RRH rental support is temporary: households must exit the program within a maximum of 2 years (sooner in many cases), at which point they are expected to pay the full rent.

Coordinated Entry. Because there are many more eligible households than can be supported with existing resources, RRH support must be rationed. Within each region (referred to as a "continuum of care" or CoC), a government-run "coordinated entry" system is responsible for matching households experiencing homelessness to non-profit providers offering RRH. This system maintains a list of households experiencing homelessness. Whenever a housing provider has the capacity to assist an additional household, it requests a “referral” from the coordinated entry system. The system then refers one of the households on its list to the provider, who reaches out to the client to verify eligibility and begin the housing search process.

Prioritization. Coordinated entry systems make referrals based on a priority queue. Each household is given a score, and whenever an opening becomes available, the highest-priority household on the list is referred to that program. The Department of Housing and Urban Development (HUD) requires that CoCs document their prioritization policies, but does not specify what these policies should be. CoCs often consider several conflicting principles when setting their policies.

- **Prioritize the most vulnerable households.** HUD does encourage CoCs to “ensure that people with the most severe service needs and levels of vulnerability are prioritized” (United States Department of Housing and Urban Development, 2015). One definition of vulnerability is given by the “Vulnerability Index – Service Prioritization Decision Assistance Tool”, or VI-SPDAT. This is a questionnaire that many CoCs ask households to complete before receiving assistance (OrgCode and CommunitySolutions, 2015). Households are then prioritized based on a “vulnerability score” calculated from their responses. Scores are higher for individuals who report chemical dependency, a history of abuse or self-harm, or recent visits to an emergency room.

- **Prioritize households most likely to remain successfully housed.** HUD asks CoCs to track and report whether households that have been supported by the Coordinated Entry system return to homelessness after receiving assistance . A CoC interested in improving this metric might seek to assist households deemed most likely to be able to support themselves after rental support expires. In practice, this might mean prioritizing households where at least one member has regular employment.

- **Prioritize households who have waited longest.** A common rationing method is to use a first in, first out (FIFO) queue or waiting list, which prioritizes individuals who have been on the list the longest.

- **Prioritize recently unhoused households.** The original intent of RRH – suggested by its name – was to assist households that only recently became homeless. A natural policy aligned with this intent is to operate a last in, first out (LIFO) queue.

Individual CoCs periodically change their prioritization policies. For example, Hennepin County used the VI-SPDAT until March 2020, when a report concluded that “use of the VI-SPDAT unfairly favors white people over people of color, thereby perpetuating racial inequities within the homeless system” (Wilkey et al., 2019). The subsequent policy prioritized the disabled, as well as chronically homeless households with the greatest time spent homelessness[1]. In August 2023, disability status

[1] Source: `https://content.govdelivery.com/accounts/MNHENNE/bulletins/280a166`

was replaced by the related but distinct concept of "medical fragility." These changes indicate that the appropriate policy is far from obvious.

1.2 Our Contributions

To tackle this problem, we formulate a dynamic model in which households transition between housed and unhoused states according to household-dependent rates. The policymaker observes the state of each household, and must choose who to assist. After receiving assistance, households immediately transition to the housed state, but remain at risk of returning to homelessness. We focus on the objective of minimizing the size of the homeless population in steady state.

Limitations of prioritizing based on vulnerability or success. Within this model, we offer definitions of vulnerability-based and success-based prioritization policies. We measure a household's vulnerability by the expected duration of an episode of homelessness if the household does not receive assistance. We measure success by the expected time that a household remains housed after receiving assistance.

We identify circumstances under which vulnerability-based and success-based policies are approximately optimal but show that in general, both approaches could result in a homeless population that is much larger than necessary (Theorem 1). The intuition is that prioritizing vulnerable households could waste resources if these households return quickly to homelessness, while prioritizing successful households is wasteful if these households were likely to find housing even without assistance.

A near-optimal policy. We propose an alternative prioritization policy that considers both vulnerability and success. This policy prioritizes households for which the harmonic mean of "expected duration of homelessness" and "expected duration of remaining housed" is largest. We show that this is near-optimal whenever there is enough RRH to assist many households (Theorem 2).

Policies based on waiting time. Implementing our proposed policy requires that the policymaker knows how long households are expected to remain unhoused (without assistance) and housed (after receiving assistance). Estimating these quantities may be difficult, and prioritizing accordingly may risk violating fair housing laws. Therefore, the second section of our paper focuses on policies that prioritize based on waiting time. These policies can be implemented with no knowledge other than when each household became homeless. We consider a variant of our model in which the policymaker chooses a distribution that specifies how long households must wait before being offered assistance.

Among policies based on waiting time, a first-in, first-out (FIFO) queue gives the most assistance to vulnerable agents (Lemma 5), and a last-in, first out (LIFO) queue gives the least (Lemma 6). As a result, if prioritizing vulnerable agents is desirable, then FIFO is optimal and LIFO is pessimal (Theorem 3). The reverse is true if prioritizing vulnerable agents is undesirable. If all households are equally vulnerable, then any policy based on waiting time is equivalent (Theorem 5). Finally, our findings continue to hold if the policymaker can offer agents a menu of waiting time distributions to choose from (Theorem 4).

1.3 Literature Review

The idea of defining vulnerability as the likelihood of experiencing long-term homelessness was proposed by Rice (2013), which developed the TAY tool to prioritize youth in need of housing support. This tool consisted of a set of questions that are used to score individuals' vulnerability. Building on this work, Rice et al. (2018) and Chan et al. (2019) both found that youth with medium levels of vulnerability have positive outcomes when matched with RRH, but youth with higher levels of vulnerability who are given RRH support are more likely to experience homelessness within the subsequent 180 days. Hsu et al. (2021) use the same measure of vulnerability, and show that it is correlated with the abandonment of RRH. These papers expose the challenge that those with the greatest difficulty of finding housing often have the lowest probability of remaining housed after receiving assistance. Brown et al. (2017) showed that 9.5% permanently housed participants re-entered the homeless service system during the follow-up period.

Other work has considered different prioritization approaches. Azizi et al. (2018) propose a mixed-integer program to prioritize housing resources for homeless youth. They take a data-driven approach with three parallel objectives in mind: efficiency, fairness and interpretability. Kube et al. (2019) propose an allocation mechanism based on predicted outcomes to reduce the number of families experiencing repeated episodes of homelessness. We complement this research by showing that prioritizing based only on success could lead to suboptimal outcomes.

Das (2022) focuses on the role of AI in allocating scarce resources. He describes three principles used by policymakers: allocate to those with the greatest need, allocate to those who will be best off after allocation and, allocate to those who would get the greatest "value added" from the resource. There is a direct match between these principles and the vulnerability-based, success-based and benefit-based policies studied in this paper. Vulnerability-based and success-based policies are also studied in Kube et al. (2022), which finds that many people consistently follow one of these approaches when asked to choose between households.

Su and Zenios (2004) study LIFO vs FIFO queues for the matching of patients to kidneys. They show that a LIFO queue leads to fewer wasted resources because it solves the misaligned incentives problem present in FIFO queues. We also establish conditions under which a LIFO queue is optimal, but the forces driving our results are quite different. In our case agents do not make decisions, and the benefits of the LIFO queue come from improved targeting (statistical discrimination), rather than strategic incentives. Also motivated by kidney allocation is the work of Nikzad and Strack (2022), which uses a model similar to the one presented in the second section of this paper to analyze the efficiency of different allocation mechanisms. That paper studies the inequality in assignment probability derived from different mechanisms. It shows that service in random order is the most equitable mechanism and, under certain conditions, it is also the most efficient. The main differences between our work and theirs are: (i) in our model agents do not depart the system and therefore agents might require multiple resources. (ii) in their model agents who leave the market unmatched are assumed to become too ill for transplant (which is a negative outcome), whereas in our case where agents could leave the queue without the policymaker's help by finding housing on their own (which is a positive outcome).

2 Discrete model with observable characteristics

There is a set $N = \{1, 2, ..., n\}$ of agents. At any moment in time, the system state is defined by a vector $\mathbf{X} \in \{0,1\}^N$: $X_i = 0$ indicates that agent i is housed, and $X_i = 1$ if agent i is unhoused. Each agent $i \in N$ is characterized by a pair of values $(\lambda_i, \mu_i) \in [\underline{\lambda}, \bar{\lambda}] \times [\underline{\mu}, \bar{\mu}] \subset \mathbb{R}_+^2$. The time agent i spends in state 0 before losing housing follows an exponential distribution with mean $1/\lambda_i$. Absent intervention, the time agent i spends in the unhoused state 1 before finding new housing follows an exponential distribution with mean $1/\mu_i$. We use $1/\mu_i$ as a measure of vulnerability: smaller μ_i correspond to more vulnerable agents.

Resources arrive according to a Poisson process with rate r. These resources allow the policymaker to move an agent from state 1 to state 0. Resources that are not used immediately are discarded. We focus on Markovian policies, where the decision of whom to help depends only on the current state. These policies can be defined by a partition of the state space $\pi = \{\pi_i\}_{i \in N}$, with $\pi_i \subseteq \{0,1\}^N$ the set of states in which the policymaker would choose to assist agent i. We require that (I) for every $i, j \in N$ with $i \neq j$ we have that $\pi_j \cap \pi_i = \emptyset$ and (II) for any $\mathbf{X} \in \pi_i$ we have $X_i = 1$. The first condition ensures that there is never ambiguity about which agent is receiving the resource. The second condition ensures that the policymaker only helps unhoused agents.

Given a policy π, the process $\mathbf{X}^\pi(t)$ is a continuous time Markov chain with a finite number of states and therefore has a unique stationary distribution. Let $\mathbf{X}^\pi$ be a random variable drawn from the steady-state distribution of $\mathbf{X}^\pi(t)$. The goal of the policymaker is to choose a policy π to minimize the expected number of unhoused agents in steady-state:

$$U(\pi) = \sum_{i=1}^{n} \mathbb{E}\left[X_i^\pi\right]. \tag{1}$$

Many of our results address *priority queues*, which are a simple type of policy that is commonly used in practice. In these policies, the policymaker has a strict ranking of agents, and always assists the highest-ranking agent. In general, the optimal policy might not be a priority queue, as shown in Appendix A.1.1. However, Theorem 2 shows that an appropriate priority queue is approximately optimal in large markets.

Definition 1. *A policy π is called a priority queue if there is some priority order $\succ$ over the agents such that $\pi_i = \{\mathbf{X} : X_i = 1$ and $X_j = 0$ for all $j \succ i\}$.*

From now on, when we consider priority queues we will assume that agents are indexed by their priority order. That is, agent 1 has the highest priority, followed by agent 2 and so on.

2.1 Vulnerability and Success based priority queues

In some cases, policymakers use vulnerability as the deciding factor for prioritizing agents. In other cases, the rate of success is the most important characteristic. We define these policies in our model as follows.

Definition 2. *A priority queue with order $\succ$* **prioritizes the vulnerable** *if $\mu_i < \mu_j \implies i \succ j$. A priority queue with order $\succ$* **prioritizes based on success** *if $\lambda_i < \lambda_j \implies i \succ j$.*

Our first result shows that there are scenarios where prioritizing based on vulnerability or success can lead to arbitrarily bad outcomes.

Theorem 1. *Let π^μ be a priority queue that prioritizes the vulnerable and π^λ a priority queue that prioritizes based on success. For any $\epsilon > 0$, there exists a set of agents N and a priority queue π^* such that*

$$U(\pi^*) \leq |N|\epsilon \tag{2}$$

$$U(\pi^\mu) \geq |N|(1-\epsilon), \tag{3}$$

and there exists a set of agents N and a priority queue π^* such that

$$U(\pi^*) \leq |N|\epsilon \tag{4}$$

$$U(\pi^\lambda) \geq |N|(1/2-\epsilon). \tag{5}$$

In our example illustrating the shortcomings of prioritizing based on vulnerability, there are a small number of vulnerable agents who have very small success rates (big λ). Prioritizing these agents wastes resources that could instead help agents with small λ find housing for the long term.

In the example illustrating the shortcomings of prioritizing based on success, there are two sets of agents of equal size. The two groups have similar success rates λ, but the group with a slightly smaller λ has a very high value of μ. As a result, resources allocated to these agents are largely wasted, as these agents were likely to find housing even without assistance. The complete proof can be found in Appendix B.1.3.

Having shown that prioritizing agents based on vulnerability or success could be arbitrarily sub-optimal, our next goal is to identify a better policy. We start by bounding the performance of any policy in Section 2.2 before defining a priority queue that incorporates both vulnerability and success in Section 2.3.

2.2 A simple lower bound on the unhoused population

Our first step is to get a clean representation of $U(\pi)$, which also gives us a lower bound on the steady-state unhoused population of any policy.

Proposition 1. *For any Markovian policy π, define*

$$r_i^\pi = r\mathbb{P}\left(\mathbf{X}^\pi \in \pi_i\right) \tag{6}$$

Then $\sum_i r_i^\pi \leq r$, and for all i, $r_i^\pi \in (0, \lambda_i)$ and $E[X_i^\pi] = \frac{\lambda_i - r_i^\pi}{\lambda_i + \mu_i}$.
Therefore, $\min_\pi U(\pi) \geq U^*$*, where*

$$U^* = \min_{r_i} \quad \sum_{i=1}^n \frac{\lambda_i - r_i}{\lambda_i + \mu_i} \tag{7}$$

$$s.t \quad 0 \leq r_i \leq \lambda_i \quad \forall i \in N \tag{8}$$

$$\sum_{i=1}^n r_i \leq r. \tag{9}$$

Proposition 1 is useful for several reasons. First, it allows us to evaluate policies based on the rates in which resources are given to each agent (r_i), instead of the more complicated partitions π. Second, it yields a useful lower bound U^* by allowing the policymaker to choose the values r_i directly (in general, this lower bound is not attainable). Third, the solution to the linear program above reveals insightful structure: it gives as many resources as possible ($r_i = \lambda_i$) to agents with the smallest values of $\lambda_i + \mu_i$. This suggests a natural priority queue which prioritizes agents based on the value $\lambda_i + \mu_i$. We study this policy in the next section.

The proof of Proposition 1 separates the state-space in three disjoint sets for every agent i: (1) H_i are all the states where agent i is housed, (2) π_i is defined as the set of states where i is unhoused and could receive a resource and, (3) Z_i are all states where i is unhoused but will not receive a resource. We then use the balance equations between these sets to compute the time that each agent spends unhoused. The complete proof is in Appendix B.1.1.

2.3 A near-optimal policy

Inspired by the linear program in Proposition 1, we will consider the policy that prioritizes agents with the lowest value of $\lambda_i + \mu_i$. We will show that this policy is approximately optimal if the number of agents and rate of resource arrival are large. Because of this result, we refer to this policy as a priority queue that prioritizes benefits.

Definition 3. *A priority queue $\pi^{\lambda+\mu}$ with order $\succ$ prioritizes benefits if and only if:*

$$\lambda_i + \mu_i < \lambda_j + \mu_j \implies i \succ j \tag{10}$$

Our next result shows that in large markets, this policy achieves an unhoused population close to the lower bound from Proposition 1.

Theorem 2. *Let π be a priority queue that prioritizes benefits. Then the difference between $U(\pi)$ and U^* is bounded by*

$$U(\pi) - U^* \leq 1 + \left(1 + \frac{1}{\underline{\lambda}}\right)\sqrt{r}. \tag{11}$$

Furthermore, if $\sum_{i\in N} \lambda_j > r$ then:

$$U(\pi) - U^* \leq 1 + \left(1 + \frac{1}{\underline{\lambda}}\right)\sqrt{\overline{\lambda} n}. \tag{12}$$

The main insight from this theorem is that the difference between a priority queue that prioritizes benefits and our lower bound grows sub-linearly in r and n. Therefore, in a market where r and n are large, the relative difference in the size of the unhoused population will be small. The proof can be found in Appendix B.1.2.

There are instances where a good performance guarantee can be made for vulnerability and success-based policies. For example, when high vulnerability is correlated with a high success rate, then prioritizing based on vulnerability will be equivalent to prioritizing based on benefit, which we already showed is asymptotically optimal. We formalize this idea in the following remark.

Remark 1. *If $\mu_i < \mu_j$ implies $\mu_i + \lambda_i < \mu_j + \lambda_j$, then prioritizing based on vulnerability is asymptotically optimal. If $\lambda_i < \lambda_j$ implies $\mu_i + \lambda_i < \mu_j + \lambda_j$ then prioritizing based on success is asymptotically optimal.*

2.4 Proof outline

To establish the result presented in Theorem 1 and Theorem 2 we use the fact that priority queues have a simple fluid approximation. The solution of the fluid approximation from Proposition 1 has a simple structure where some agents receive all the resources and are always housed, another set of agents does not receive any resources and are housed only a fraction of the time and, there is one agent that receives some resources but not enough to be constantly housed. We will have a similar structure for the fluid approximation of general priority queues. We will use x_i^π to represent the fluid approximation of a priority queue π:

$$x_i^\pi = \begin{cases} 0 & \text{if } \sum_{j=1}^{i} \lambda_j < r \\ \frac{\sum_{j=1}^{i} \lambda_j - r}{\lambda_i + \mu_i} & \text{if } \sum_{j=1}^{i-1} \lambda_j \leq r \leq \sum_{j=1}^{i} \lambda_j \\ \frac{\lambda_i}{\lambda_i + \mu_i} & \text{o.w} \end{cases} \tag{13}$$

The goal of this section is to show that this fluid approximation is, forgive the redundancy, a good approximation of the discrete model. To this end, we use Lemma 1 (proved in Weng et al. (2020)) multiple times to upper bound the difference between the expected number of unhoused agents and its fluid approximation. This result uses a Lyapunov function from the state space to

$\mathbb{R}_+$ and shows that under certain conditions the probability that the Lyapunov function is big falls geometrically.

Lemma 1 (Weng et. al Lemma 6)**.** *Consider a continuous time Markov Chain* $\{X(t) : t \geq 0\}$ *on a finite state space* $\mathcal{X}$*. Assume it has a unique stationary distribution. For a Lyapunov function* $V : \mathcal{X} \to \mathbb{R}_+$ *define* $GV(x) = \sum_{x' \in \mathcal{X}} r_{x,x'}(V(x') - V(x))$ *where* $r_{x,x'}$ *is the transition rate from* x *to* x'*. Suppose that*

$$\nu_{max} = \sup_{x,x' \in \mathcal{X} : r_{x,x'} > 0} |V(x') - V(x)| < \infty \tag{14}$$

$$f_{max} = \max\left\{0, \sup_{x \in \mathcal{X}} \sum_{x' : V(x') > V(x)} r_{x,x'}(V(x') - V(x))\right\} < \infty \tag{15}$$

Given a set $\mathcal{E}$*. If for some* $B > 0$, $\gamma > 0$, $\xi \geq 0$ *it holds:* (1) $GV(x) < -\gamma$ *when* $V(x) \geq B$ *and* $x \in \mathcal{E}$*.* (2) $GV(x) < \xi$ *when* $V(x) \geq B$ *and* $x \notin \mathcal{E}$*. Then for all positive integers* m*, if* X^* *is the steady-state random variable, it holds that*

$$\mathbb{P}(V(X^*) \geq B + 2\nu_{max} j) \leq \left(\frac{f_{max}}{f_{max} + \gamma}\right)^j + \left(\frac{\xi}{\gamma} + 1\right)\mathbb{P}(x \notin \mathcal{E}). \tag{16}$$

The general bound is given in the following proposition:

Proposition 2. *Let* π *be a priority queue and choose an* $\alpha \in (1/2, 1)$*. The distance between the original problem and the fluid approximation is less than:*

$$\sum_{i \in N} |\mathbb{E}[X_i^\pi] - x_i^\pi| \leq 2r^{1-\alpha} + \frac{2r^\alpha}{\underline{\mu} + \underline{\lambda}} + \frac{r}{\underline{\mu} + \underline{\lambda}} exp\left\{-\frac{r^{2\alpha}}{8(\bar{\lambda} + \bar{\mu})(r + r^\alpha)(1 + \bar{\mu}/\underline{\lambda})}\right\} \tag{17}$$

To show this result we split agents into 3 disjoint sets of agents $N_1 \cup N_2 \cup N_3 = N$. The first one is a high-priority group that is likely to be housed. We use a $\alpha \in (1/2, 1)$ to determine this group: $\sum_{i \in N_1} \lambda_i = r - r^\alpha$. We can use Lemma 2 to find the difference between this group's utility and the fluid approximation. The second group should be small enough that it should have little impact compared to the overall population. The last group consists of agents that are unlikely to get help from the policymaker: $\sum_{i \in N_1 \cup N_2} \lambda_i = r + r^\alpha$. We use Lemma 3 to upper-bound the distance between this group and its fluid approximation.

To bound the difference of the high-priority group that is likely to be housed N_1 we use the following Lyapunov function that for every state $\mathbf{X}$ counts the number of unhoused agents in the set M:

$$V(\mathbf{X}) = \sum_{i \in M} X_i. \tag{18}$$

The result for this first group is summarized in Lemma 2.

Definition 4. *A policy is non-wasteful if a resource will only be discarded if, at the time of its arrival, everyone is housed:* $\cup_{i=1}^{n} \pi_i = \{0,1\}^N \setminus \{0\}^N$.

Lemma 2. *Let* $M \subseteq N$ *be a subset of agents and let* π *be a non-wasteful policy such that any agent* $j \notin M$ *could receive a resource only if everyone in* M *is already housed: for all* $\mathbf{X} \in \pi_j$ *we have* $X_i = 0$ *for all* $i \in M$*. If* $\sum_{i \in M} \lambda_i \leq r$ *then:*

$$\mathbb{P}\left(\sum_{i \in M} X_i^{\pi} \geq 2k+1\right) \leq \left(\frac{\sum_{i \in M} \lambda_i}{r}\right)^k. \tag{19}$$

and

$$\mathbb{E}\left[\sum_{i \in M} X_i^{\pi}\right] \leq \frac{2r}{r - \sum_{i \in M} \lambda_i}. \tag{20}$$

For the case of agents that are unlikely to get help from the policymaker (N_3), we bound the difference from the fluid approximation in the following result.

Lemma 3. *Take an* $\alpha \in (0,1)$ *and let* $M \subseteq N$ *be a subset of agents such that* $r^{\alpha} = \sum_{i \in M} \lambda_i - r \geq 4(\bar{\lambda} + \bar{\mu})$*. Let* π *be a non-wasteful policy such that any agent* $j \notin M$ *could receive a resource only if everyone in* M *is already housed: for all* $\mathbf{X} \in S_j$ *we have* $X_i = 0$ *for all* $i \in M$*.*

$$\sum_{i \in N \setminus M} \left|\mathbb{E}\left[X_i^{\pi}\right] - x_i^{\pi}\right| \leq \frac{r}{\underline{\mu} + \underline{\lambda}} exp\left\{-\frac{r^{2\alpha}}{8(\bar{\lambda} + \bar{\mu})(r + r^{\alpha})(1 + \bar{\mu}/\underline{\lambda})}\right\} \tag{21}$$

3 Prioritization based on waiting time

In this section, we study a variant of the model from the previous section. As before, we have a set $N = \{1, 2, ..., n\}$ of agents, which move between housed and unhoused states. We let $X_i(t)$ denote the state of agent i at time t: $X_i(t) = 0$ if i is housed, and $X_i(t) = 1$ if i is unhoused. In the previous section, the policymaker selected a policy that specified which agent to help when a resource arrives. Each policy induced a (random) waiting time that each agent needed to spend unhoused before receiving help from the policymaker. In this section, we empower the policymaker to directly choose the waiting time distribution for each agent. This decouples the evolution of the agents from each other, allowing us to study them independently.

Formally, we consider *interventions* parameterized by distributions $\{\mathbf{F}_i\}_{i \in N}$. Given an intervention, the evolution of X_i is as follows. Every time agent i enters state 1 (becomes *unhoused*), the policymaker draws a random variable T_i from cumulative distribution function $\mathbf{F}_i$. If agent i remains unhoused for T_i periods, the policymaker intervenes by providing housing to i.

As in Section 2, we evaluate interventions based on the steady-state size of the unhoused population. Formally, let $H_i(t) = \int_0^t 1 - X_i(t)dt$ be a random variable that keeps track of the amount of time agent i has spent housed up to time t, and define

$$x_i^F = 1 - \lim_{t\to\infty} \frac{H_i(t)}{t} \tag{22}$$

$$U(\mathbf{F}) = \sum_{i\in N} x_i^F. \tag{23}$$

We will use $U(\mathbf{F})$ to compare different interventions. It is trivial to see that $U(\mathbf{F})$ is minimized by choosing the intervention consisting of a point mass at zero (that is, help every agent immediately).

To incorporate resource constraints, we will only compare policies that require similar rates of resource consumption, defined as follows. Let $N_i(t)$ be a renewal process that counts the number of times agent i has received help from the designer up to time t. For any intervention $\mathbf{F}$, define

$$r_i^F = \lim_{t\to\infty} \frac{N_i(t)}{t} \tag{24}$$

$$r^F = \sum_{i\in N} r_i^F. \tag{25}$$

The quantity r^F gives the rate at which resources must be used to implement intervention $\mathbf{F}$.

The main result of this section states conditions in which *FIFO* and *LIFO* queues are optimal and pessimal or vice versa. We build to this by first defining targeting and majorization. We then define anonymous interventions and show that *FIFO* queues target the most vulnerable and *LIFO* queues target the least vulnerable. With all the building blocks, we formally introduce the main result in Theorem 3. We conclude this section by introducing a broader family of interventions where the policymaker builds a menu of interventions and lets the agents choose between them. We show that even in this larger set of interventions, under the conditions stated in Theorem 3, the *FIFO* and *LIFO* queues are optimal and pessimal or vice versa.

3.1 Targeting through interventions

Similar to the model from the previous sections, we can write the unhoused population as a function of the rate resources used in each agent under intervention $\mathbf{F}$ as follows (see Corollary 1):

$$U(\mathbf{F}) = \sum_{i\in N} \frac{\lambda_i - r^F}{\lambda_i + \mu_i} \tag{26}$$

Note that interventions are more flexible than the class of policies considered in Section 2, in the sense that we can construct interventions that lead to any point inside the feasible region of the LP presented in Proposition 1. For example, the policymaker achieves $r_i = 0$ by never allocating to agent i ($\mathbf{F}_i$ is a point mass at ∞), and achieves $r_i = \lambda_i$ by allocating to agent i immediately ($\mathbf{F}_i$ is a point mass at 0). Any desired rate of resource use $r_i^F \in [0, \lambda_i]$ can also be achieved by a suitably chosen intervention $\mathbf{F}_i$.

We use the concept of majorization to compare interventions.

Definition 5. *Intervention* $\mathbf{F}$ *majorizes intervention* $\mathbf{G}$ *up to agent* k *(denoted* $\mathbf{F} \succeq^k \mathbf{G}$*) if and* *only if for all $j \leq k$,*

$$\sum_{i=1}^{j} r_i^F \geq \sum_{i=1}^{j} r_i^G \tag{27}$$

If $\mathbf{F}$ *majorizes* $\mathbf{G}$ *up to agent* n *we simply say that* $\mathbf{F}$ *majorizes* $\mathbf{G}$ *and write it as* $\mathbf{F} \succeq \mathbf{G}$.

Suppose agents are ordered from most preferred to least preferred to get help from the policymaker. An intervention majorizes another intervention if more resources are used in the more preferred agents. With this definition, we show that if agents are ordered from smallest $\lambda_i + \mu_i$ to biggest then if intervention $\mathbf{F}$ majorizes intervention $\mathbf{G}$, the former leads to a smaller unhoused population: $U(\mathbf{F}) \leq U(\mathbf{G})$.

Lemma 4. *Suppose* $\mu_i + \lambda_i \leq \mu_j + \lambda_j$ *for every* $i \leq j$. *If intervention* $\mathbf{F}$ *majorizes intervention* $\mathbf{G}$ *then* $U(\mathbf{F}) \leq U(\mathbf{G})$.

3.2 Anonymous interventions

Our analysis will focus on *anonymous* interventions that treat all agents equally.

Definition 6. *An intervention is* **anonymous** *if* $\mathbf{A}_i = A$ *for all agents.*

This is partially motivated by the fact that agents' characteristics could be difficult to estimate. Furthermore, even if vulnerability and success could be accurately measured, the policymaker could have fairness constraints that prevent her from using these to prioritize agents. In what follows, we sometimes use the word "intervention" to refer to an anonymous intervention.

Two commonly used anonymous interventions are the *FIFO* and *LIFO* queues. We define these interventions as follows.

Definition 7. *An anonymous intervention* $\mathbf{F}_t$ *is called a FIFO Queue if there is some* t *such that:*

$$\mathbf{F}_t(x) = \begin{cases} 0 & \text{if } x \leq t \\ 1 & \text{if } x > t \end{cases} \tag{28}$$

Definition 8. *An anonymous intervention* $\mathbf{L}_q$ *is called a LIFO Queue if there is some* q *such that:*

$$\mathbf{L}_q(x) = q, \quad \forall x \geq 0. \tag{29}$$

These definitions match those given by Nikzad and Strack (2022). They can be thought of as large market limits of discrete FIFO and LIFO queues. Note that our definition of a FIFO queue maintains the property that whenever an agent receives assistance from the policymaker, it is the agent who has been unhoused for the longest time. Similarly, our definition of a LIFO queue ensures that whenever an agent receives assistance, it is the agent who has most recently become unhoused.

3.3 Targeting the vulnerable

We next show that a *FIFO Queue* is the anonymous intervention that best targets the most vulnerable.

Lemma 5. *Suppose* $\mu_i \leq \mu_j$ *for every* $i \leq j$*. Let* $\mathbf{A}$ *be any anonymous intervention. There exists a FIFO Queue* $\mathbf{F}_t$ *that majorizes* $\mathbf{A}$ *and uses the same number of resources* $r^A = r^{F_t}$

Note that $\mu_i \leq \mu_j$ is a labeling of the agents and not a condition of the primitives of the model. This lemma says that we can always choose a *FIFO* queue that gives more resources to agents with lower μ_i. The intuition behind this result is that the longer you wait to help an agent the more vulnerable this agent is likely to be. The proof of this result can be found in Appendix B.2.2.

On the other hand, a *LIFO Queue* is the worst anonymous intervention at targeting vulnerable agents.

Lemma 6. *Suppose* $\mu_i \leq \mu_j$ *for every* $i \leq j$*. Let* $\mathbf{A}$ *be any anonymous intervention. There exists an LIFO Queue* $\mathbf{L}_q$ *that is majorized by* $\mathbf{F}$ *and uses the same number of resources* $r^A = r^{L_q}$

Because time spent waiting correlates with μ_i, eliminating waiting time removes all relations between μ_i and resource allocation. Any other intervention will lead to agents with smaller μ_i receiving more resources. The proof of this lemma is in Appendix B.2.3.

Lemma 5 and Lemma 6 show that *FIFO* and *LIFO* queues target the most and least vulnerable, respectively. If everyone is equally vulnerable ($\mu_i = \mu$ for all $i \in N$) then any two anonymous interventions that use the same rate of resources will lead to the same outcome for every agent. Details of this result are shown in Appendix A.3.

3.4 Optimality of FIFO and LIFO queues

Theorem 3 formalizes the central result of this section, which is that if the most vulnerable agents have the smallest $\lambda_i + \mu_i$ then a *FIFO Queue* is the best anonymous intervention. Conversely, if the most vulnerable agents have the largest $\lambda_i + \mu_i$ then a *LIFO Queue* is the best anonymous intervention.

Theorem 3. *Let* $\mathbf{F}_t$ *be a FIFO Queue such that* $r^{F_t} = r$*. Let* $\mathbf{L}_q$ *be a LIFO Queue such that* $r^{L_q} = r$*. Finally, let* $\mathbf{A}$ *be any anonymous intervention such that* $r^A = r$*. If*

$$\mu_i < \mu_j \implies \mu_i + \lambda_i < \lambda_j + \mu_j, \tag{30}$$

then:

$$U(\mathbf{F_t}) \leq U(\mathbf{A}) \leq U(\mathbf{L_q}). \tag{31}$$

On the other hand, if

$$\mu_i < \mu_j \implies \mu_i + \lambda_i > \lambda_j + \mu_j, \tag{32}$$

then the inequalities reverse:

$$U(\mathbf{F}_t) \geq U(\mathbf{A}) \geq U(\mathbf{L}_q). \tag{33}$$

Note that the condition in (30) corresponds exactly to the scenario in which targeting the most vulnerable agents is approximately optimal (see Remark 1). In this case, Theorem 3 states that FIFO is the optimal anonymous policy. Meanwhile, condition (32) is readily shown to imply that success-based policies prioritize agents with the lowest $\lambda_i + \mu_i$. Therefore, in circumstances where Theorem 3 guarantees that LIFO is the best anonymous policy, the policymaker would like to use a success-based prioritization policy if this were possible.

3.5 Menu of interventions

Our focus on anonymous interventions is partly motivated by the idea that the parameters μ_i and λ_i may not be observable to the policymaker. In many mechanism design settings with private information, it is possible to partially elicit this information by offering choices to agents. This motivates us to consider an extension in which unhoused agents can choose among several possible interventions. For example, agents could be asked to choose between two programs: one that allocates resources on a FIFO basis, and another that allocates each new resource by lottery.

To capture such possibilities, we consider an extension in which the policymaker offers a menu of anonymous interventions $\mathcal{M} = \{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, .., \mathbf{A}^{(m)}\}$ to each agent. Let $p(\mu, \mathbf{A})$ be the probability that an agent with with housing rate μ receives assistance under intervention $\mathbf{A}$:

$$p(\mu, \mathbf{A}) = \mathbb{P}_{Y\sim\mathbf{A}}(Y \leq Z_\mu), \tag{34}$$

where $Z_\mu \sim Exp(\mu)$. We assume that each agent i will choose the intervention that maximizes the probability of getting help:

$$c_i(\mathcal{M}) \in \arg\max_{\mathbf{A}\in\mathcal{M}} \{p(\mu_i, A)\} \tag{35}$$

In our model, this is equivalent to choosing the intervention that minimizes the agent's steady-state fraction of time spent unhoused.

Given that we are allocating a homogeneous resource, in a setting where all agents want to maximize their probability of receiving the resource and payments are not allowed, one might expect that agents will make identical choices from the menu.

This intuition turns out to be incorrect. In Appendix A.2.1 we present an example where the policymaker can induce different agents to make different choices. Furthermore, offering a menu of interventions achieves a strictly better objective value (fewer unhoused agents) than can be achieved by any single anonymous intervention that uses the same number of resources as the menu.

Although using menus can sometimes be beneficial for the policymaker, we show that under the conditions identified in Theorem 3, a *FIFO Queue* is still the optimal (pessimal) policy and a *LIFO Queue* is the pessimal (optimal) policy. The detailed result is presented in Appendix A.2.

4 Conclusions

Discussion of assumptions.

- **Invariance of housing source**. In our stylized model, we assume that the time an agent will spend housed is independent of whether the housing was found independently or if it was an RRH resource allocated by the policymaker. In practice, RRH is a form of assistance that helps subsidize housing and comes with help from a case worker. This means that a household that receives an RRH should have better support than if they found housing independently.

- **Immediate housing**. In practice, receiving RHH does not mean that a household will be housed immediately since an RHH is a rent subsidy, but it is still the household's responsibility (with help from a caseworker) to find housing. This process could take a long time. In our model, we assume that an agent that gets allocated a resource will transition to housed immediately.

Directions for Future Work

- **Heterogeneous Resources.** Most coordinated entry systems oversee two primary forms of housing support: rapid rehousing (RRH) and permanent supportive housing (PSH). In this paper, we focused on modeling (RRH), which is meant to be a fast way to move people out of homelessness. Permanent Supportive Housing, by contrast, is intended to provide indefinite rental support and services such as mental health counseling, substance abuse treatment, childcare, employment training, and more. As such, it is more expensive to provide than RRH, and always in short supply. A natural extension of our model would be to assume that individuals who receive PSH have much lower rates of returning to homelessness (lower λ), and to optimize the joint allocation of these two different types of resources.

- **Alternative Social Welfare Functions.** Our work assumes that the goal of the policymaker is to minimize the total number of households experiencing homelessness. The policies that achieve this goal may entirely abandon the most vulnerable members of the population, which could be perceived as undesirable. Future work could investigate other social welfare functions that incorporate fairness across individuals, such as Nash social welfare (maximizing the product of household utilities) or Rawlsian social welfare (maximizing the utility of the worst-off household). Finally, we could have different welfare weights for different households. For example, a policymaker might prefer to help a family rather than a single individual.

References

Arnosti, N. and Shi, P. (2020). Design of lotteries and wait-lists for affordable housing allocation. *Management Science*, 66(6):2291–2307.

Azizi, M. J., Vayanos, P., Wilder, B., Rice, E., and Tambe, M. (2018). Designing fair, efficient, and interpretable policies for prioritizing homeless youth for housing resources. In *Integration of Constraint Programming, Artificial Intelligence, and Operations Research: 15th International Conference, CPAIOR 2018, Delft, The Netherlands, June 26–29, 2018, Proceedings 15*, pages 35–51. Springer.

Brown, M., Vaclavik, D., Watson, D. P., and Wilka, E. (2017). Predictors of homeless services re-entry within a sample of adults receiving homelessness prevention and rapid re-housing program (hprp) assistance. *Psychological services*, 14(2):129.

Chan, H., Rice, E., Vayanos, P., Tambe, M., and Morton, M. (2019). From empirical analysis to public policy: Evaluating housing systems for homeless youth. In *Machine Learning and Knowledge Discovery in Databases: European Conference, ECML PKDD 2018, Dublin, Ireland, September 10–14, 2018, Proceedings, Part III 18*, pages 69–85. Springer.

Das, S. (2022). Local justice and the algorithmic allocation of scarce societal resources. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 36, pages 12250–12255.

Hsu, H.-T., Hill, C., Holguin, M., Petry, L., McElfresh, D., Vayanos, P., Morton, M., and Rice, E. (2021). Correlates of housing sustainability among youth placed into permanent supportive housing and rapid re-housing: A survival analysis. *Journal of Adolescent Health*, 69(4):629–635.

Kube, A., Das, S., and Fowler, P. J. (2019). Allocating interventions based on predicted outcomes: A case study on homelessness services. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 33, pages 622–629.

Kube, A., Das, S., Fowler, P. J., and Vorobeychik, Y. (2022). Just resource allocation? how algorithmic predictions and human notions of justice interact. In *Proceedings of the 23rd ACM Conference on Economics and Computation*, pages 1184–1242.

Nikzad, A. and Strack, P. (2022). Equity in dynamic matching: Extreme waitlist policies. *Available at SSRN*.

OrgCode and CommunitySolutions (2015). Vulnerability index - service prioritization decision assistance tool, version 2.0.

Rice, E. (2013). The tay triage tool: A tool to identify homeless transition age youth most in need of permanent supportive housing. *New York: The Corporation for Supportive Housing http://www. csh. org/wp-content/uploads/2014/02/TAY_TriageTool_2014. pdf*.

Rice, E., Holguin, M., Hsu, H.-T., Morton, M., Vayanos, P., Tambe, M., and Chan, H. (2018). Linking homelessness vulnerability assessments to housing placements and outcomes for youth. *Cityscape*, 20(3):69–86.

Su, X. and Zenios, S. (2004). Patient choice in kidney allocation: The role of the queueing discipline. *Manufacturing & Service Operations Management*, 6(4):280–301.

United States Department of Housing and Urban Development (2015). Coordinated entry policy brief.

Weng, W., Zhou, X., and Srikant, R. (2020). Optimal load balancing in bipartite graphs. *arXiv preprint arXiv:2008.08830*.

Wilkey, C., Donegan, R., Yampolskaya, S., and Cannon, R. (2019). Coordinated entry systems racial equity analysis of assessment data. *Needham, MA: C4 Innovations*.

Appendix A Examples and extensions

A.1 Results on Optimal Policies

A.1.1 The optimal policy may not be a priority queue

Example 1. *Three agents with* $\lambda_1 = 1.08, \lambda_2 = 1.8, \lambda_3 = 1.68$ *and* $\mu_1 = 1.8, \mu_2 = 1.01, \mu_3 = 1.93$. *Rate of resources* $r = 0.94$.

The optimal policy in example Example 1 is

$$\pi_1 = \{(1,0,0),(1,1,0),(1,0,1)\} \tag{36}$$

$$\pi_2 = \{(0,1,0),(0,1,1),(1,1,1)\} \tag{37}$$

$$\pi_3 = \{(0,0,1)\} \tag{38}$$

Note that this policy is not a priority queue because the decision to help agent 1 or agent 2 depends on whether agent 3 is unhoused. Specifically, if only agents 1 and 2 are unhoused, then 1 will get the resource, but if all three agents are unhoused then agent 2 receives help.

A.1.2 The optimal policy depends on the rate of resource arrival

In Example 1 if the rate of resources is $r = 1.5$ then the optimal policy is a priority queue with $2 \succ 1 \succ 3$. This prioritizes based on $\lambda_i + \mu_i$. Note that this is different from the policy describe in the previous section, despite involving the same set of agents.

A.2 Menus of interventions

We now present results for the case where the policymaker allows agents to choose from a menu of anonymous interventions $\mathcal{M} = \{F_1, \dots, F_m\}$. We let $c_i(\mathcal{M})$ denote the choice made by agent i, let $r^{\mathcal{M}}$ denote the number of resources required to implement menu $\mathcal{M}$, and $U(\mathcal{M})$ denote the associated expected size of the unhoused population:

$$r^{\mathcal{M}} = \sum_{i=1}^{n} r_i^{c_i(\mathcal{M})} \tag{39}$$

$$U(\mathcal{M}) = \sum_{i=1}^{n} x_i^{c_i(\mathcal{M})}. \tag{40}$$

Theorem 4. *Let* $\mathbf{F}_t$ *be a FIFO Queue such that* $r^{F_t} = r$*. Let* $\mathbf{L}_q$ *be a LIFO Queue such that* $r^{L_q} = r$*. Finally, let* $\mathcal{M} = \{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, ..., \mathbf{A}^{(m)}\}$ *be a menu of anonymous interventions such that* $r^{\mathcal{M}} = r$*. If*

$$\mu_i < \mu_j \implies \mu_i + \lambda_i < \lambda_j + \mu_j, \tag{41}$$

then:

$$U(\mathbf{F_t}) \leq U(\mathcal{M}) \leq U(\mathbf{L}_q). \tag{42}$$

On the other hand, if

$$\mu_i < \mu_j \implies \mu_i + \lambda_i > \lambda_j + \mu_j, \tag{43}$$

then:

$$U(\mathbf{F}_t) \geq U(\mathcal{M}) \geq U(\mathbf{L}_q). \tag{44}$$

The proof of this result follows the same steps as Theorem 3, and uses the following results. Note that Lemma 7 generalizes Lemma 5 (*FIFO* queues target the most vulnerable) and Lemma 8 generalizes Lemma 6 (*LIFO* queues target the least vulnerable).

Lemma 7. *Suppose $\mu_i \leq \mu_j$ for every $i \leq j$. Let $\mathcal{M} = \{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, .., \mathbf{A}^{(m)}\}$ be a menu of anonymous interventions, then there exists a FIFO Queue $\mathbf{F}_t$ that majorizes $\mathcal{M}$ and uses the same number of resources $r^{\mathcal{M}} = r^{F_t}$*

Lemma 8. *Suppose $\mu_i \leq \mu_j$ for every $i \leq j$. Let $\mathcal{M} = \{\mathbf{A}^{(1)}, \mathbf{A}^{(2)}, .., \mathbf{A}^{(m)}\}$ be a menu of anonymous interventions, then there exists a LIFO Queue $\mathbf{L}_q$ that is majorized by $\mathcal{M}$ and uses the same number of resources $r^{\mathcal{M}} = r^{L_t}$*

The proofs of these results can be found in Appendix B.2.7.

A.2.1 Example where a menu of interventions improves outcome

Although Theorem 4 identifies conditions under which the optimal menu involves a single intervention (either a FIFO or LIFO queue), this is not generally the case. The following example demonstrates that a menu could lead to a lower unhoused population than any single intervention that uses the same number of resources.

Example 2. *There are three agents with paramaters $\mu_1 = 1, \mu_2 = 2, \mu_3 = 3$ and $\lambda_1 = 3, \lambda_3 = 1$ and $\lambda_2 >> \lambda_1$. The menu $\mathcal{M} = \{\mathbf{L}_q, \mathbf{F}_t\}$ with $q = 0.2$ and $t = -\frac{log(q)}{\mu_2} \approx 0.8$.*

A graphical representation of this example can be found in Fig. 1.

Proposition 3. *For Example 2, any anonymous intervention that leads to the same unhoused population as $\mathcal{M}$ must use resources at a greater rate.*

Proof. In Example 2, a designer would want to help agents 1 and 3 as much as possible and avoid agent 2, since $\lambda_1 + \mu_1 = \lambda_3 + \mu_3 << \lambda_2 + \mu_2$. Note that agent 2 is indifferent between either option in the menu. Agent 1 strictly prefers $\mathbf{F}_t$ and agent 3 prefers $\mathbf{L}_q$. The first important thing to note is that there is no single item menu that can achieve the same allocation: There is no anonymous $\mathbf{A}$ such that $x_i^A = x_i^{\mathcal{M}}$ for all i. This comes from noting that

$$p(\mu_1, c_1(\mathcal{M})) > p(\mu_2, c_2(\mathcal{M})) = p(\mu_3, c_3(\mathcal{M})). \tag{45}$$

More importantly, a single-item menu that achieves the same utilitarian welfare will necessarily use more resources. To see why this is the case let $\mathbf{A}$ be an anonymous intervention such that $x_1^A + x_2^A + x_3^A = x_1^{\mathcal{M}} + x_2^{\mathcal{M}} + x_3^{\mathcal{M}}$. Because $\mathbf{A}$ is an anonymous intervention one of the following must be true $x_1^A < x_1^{\mathcal{M}}$, $x_2^A < x_2^{\mathcal{M}}$ or $x_3^F < x_3^{\mathcal{M}}$. We will show that $x_2^F < x_2^{\mathcal{M}}$.

- If $x_1^A < x_1^{\mathcal{M}}$ then $p(\mu_1, \mathbf{A}) > p(\mu_1, c_2(\mathcal{M}))$ and therefore from Lemma 12 and the fact that $c_1(\mathcal{M}) = \mathbf{F}_t$ is a *FIFO Queue* we get:

  $$p(\mu_2, \mathbf{A}) \geq p(\mu_1, \mathbf{A})^2 > p(\mu_1, c_1(\mathcal{M}))^2 = p(\mu_2, c_2(\mathcal{M})). \tag{46}$$

  This implies that $x_2^F < x_2^{\mathcal{M}}$.

- If $x_3^F < x_3^{\mathcal{M}}$ we get that $p(\mu_3, \mathbf{A}) > p(\mu_3, c_3(\mathcal{M}))$ and therefore from Lemma 12 we get $p(\mu_2, \mathbf{A}) > p(\mu_2, c_2(\mathcal{M}))$ which implies that $x_2^A < x_2^{\mathcal{M}}$.

We conclude that $x_2^A < x_2^{\mathcal{M}}$. Note that we can write the difference in resources used as follows:

$$\begin{aligned}
r_1^{\mathcal{M}} + r_2^{\mathcal{M}} + r_3^{\mathcal{M}} - r_1^A - r_2^A - r_3^A &= (x_1^A - x_1^{\mathcal{M}})(\lambda_1 + \mu_1) + (x_2^A - x_2^{\mathcal{M}})(\lambda_2 + \mu_2) && (47)\\
&\quad + (x_3^A - x_3^{\mathcal{M}})(\lambda_3 + \mu_3) && (48)\\
&= (x_1^A - x_1^{\mathcal{M}} + x_2^A - x_2^{\mathcal{M}} + x_3^A - x_3^{\mathcal{M}})(\lambda_1 + \mu_1) && (49)\\
&\quad + (-x_2^A + x_2^{\mathcal{M}})(\lambda_1 + \mu_1) + (x_2^A - x_2^{\mathcal{M}})(\lambda_2 + \mu_2) && (50)\\
&= (x_2^A - x_2^{\mathcal{M}})(\lambda_2 + \mu_2 - \lambda_1 - \mu_1) && (51)\\
&< 0 && (52)
\end{aligned}$$

where Eq. (49) comes from $\lambda_1 + \mu_1 = \lambda_3 + \mu_3$. Therefore any single-item menu will have to use more resources to get the same utilitarian welfare.

□

A.3 Equivalence of interventions when agents are equally vulnerable

The final result of this section shows that if all agents are equally vulnerable ($\mu_i = \mu$ for all i) then all anonymous interventions that use the same number of resources will lead to the same outcome for every agent.

Theorem 5. *Let $\mathbf{F}$ and $\mathbf{G}$ be two anonymous inteventions such that $r^F = r^G$. Suppose that $\mu_i = \mu$ for all agents. Then $x_i^F = x_i^G$ for all agents i.*

Because an anonymous intervention only differentiates between agents based on the time they wait for help having the same μ means that all agents have the same probability of getting help.

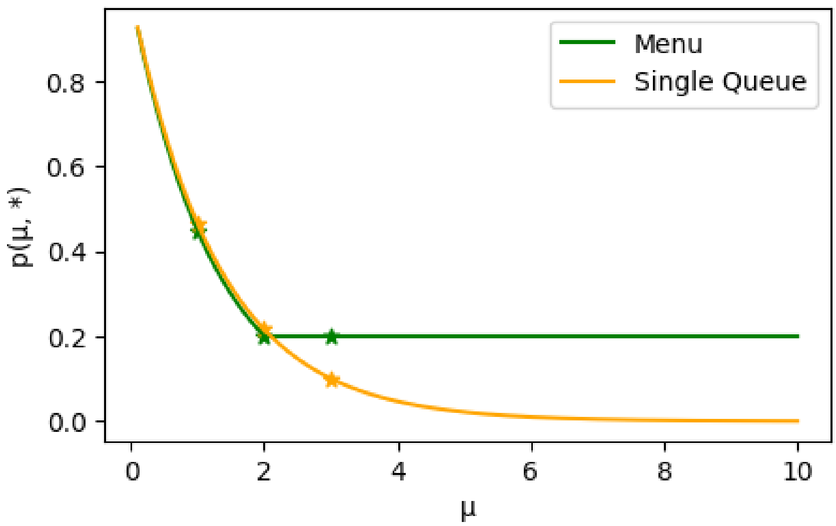

Figure 1: The probability of getting help $p(\mu, \bullet)$ vs μ for the menu $\mathcal{M}$ from Example 2 and a *FIFO Queue* that uses the same number of resources. The $\star$ shows the probability for agents in Example 2.

This is true for all anonymous interventions. It is also the case that the number of resources used by an agent has a one-to-one correspondence to the probability of getting help. Therefore, if two anonymous interventions use the same number of resources they must also lead to the same probability of getting help for every agent, which is equivalent to leading to the same outcome.

This result closely relates to Theorem 1 in Arnosti and Shi (2020), which shows the equivalence of different allocation mechanisms under the assumption of heterogeneous departure rates. Departures in their work correspond to unhoused individuals finding housing without help in ours.

Appendix B Proofs

B.1 Discrete model with observable characteristics: proofs and lemmas

B.1.1 Proof of Proposition 1

Proof. Let $H_i \subset \{0,1\}^N$ be the set of states where agent i is housed. Also, let $Z_i = \{0,1\}^N \setminus (H_i \cup \pi_i)$ be the set of states where agent i is housed but would not get a resource from the policy-maker. Note that $H_i \cap Z_i = \emptyset$, $\pi_i \cap Z_i = \emptyset$ and $H_i \cap \pi_i = \emptyset$ For every agent $i \in N$ we can write the following balance equation:

$$\lambda_i \mathbb{P}\left(\mathbf{X}^\pi \in H_i\right) = \mu_i \mathbb{P}\left(\mathbf{X}^\pi \in Z_i\right) + (\mu_i + r)\mathbb{P}\left(\mathbf{X}^\pi \in \pi_i\right) \tag{53}$$

Because $H_i \cup \pi_i \cup Z_i = \{0,1\}^n$ is the entire state space we can write the balance equation, cancel like terms, and solve for the probability of being in H_i:

$$\lambda_i \mathbb{P}(\mathbf{X}^\pi \in H_i) = \mu_i(1 - \mathbb{P}(\mathbf{X}^\pi \in H_i) - \mathbb{P}(\mathbf{X}^\pi \in \pi_i)) + (\mu_i + r)\mathbb{P}(\mathbf{X}^\pi \in \pi_i) \quad (54)$$

$$(\lambda_i + \mu_i)\mathbb{P}(\mathbf{X}^\pi \in H_i) = \mu_i + r\mathbb{P}(\mathbf{X}^\pi \in \pi_i) \quad (55)$$

$$\mathbb{P}(\mathbf{X}^\pi \in H_i) = \frac{\mu_i + r_i^\pi}{\lambda_i + \mu_i} \quad (56)$$

Where the last equality comes from the definition of r_i. Note that $\mathbb{E}[X_i^\pi] = 1 - \mathbb{P}(\mathbf{X}^\pi \in H_i)$ and therefore:

$$E[X_i^\pi] = \frac{\lambda_i - r_i^\pi}{\lambda_i + \mu_i} \quad (57)$$

We can write the policymakers problem as follows:

$$\min_\pi \ U(\pi) = \min_\pi \ \sum_{i=1}^n \frac{\lambda_i - r_i^\pi}{\lambda_i + \mu_i} \quad (58)$$

$$\text{s.t } r_i^\pi = r\mathbb{P}(\mathbf{X}^\pi \in \pi_i), \quad \forall i \in N \quad (59)$$

We can relax this problem using r_i's as decision variables. First, using the fact that the sets π_i are disjoint we get an upper bound on the sum of r_i's

$$\sum_{i=1}^n r_i^\pi = r \sum_{i=1}^n \mathbb{P}(\mathbf{X}^\pi \in \pi_i) \leq r \quad (60)$$

Also, because of 56 and the fact that a probability is always less than or equal to 1 we have that $r_i^\pi \leq \lambda_i$. Finally, by the definition of r_i^π we must have that $r_i^\pi \geq 0$. Therefore, we can relax constraint 59 with the following set of constraints:

$$0 \leq r_i^\pi \leq \lambda_i \quad (61)$$

$$\sum_{i=1}^n r_i^\pi \leq r \quad (62)$$

and optimize based on the rate of resources r_i^π. □

B.1.2 Proof of Theorem 2

Proof. First note that $U^* = \sum_{i \in N} x_i^\pi$. Let's call the agent that receives some resources but not enough to make him constantly house the pivot agent, we will use i^* to refer to this agent. We can therefore write the difference in utility as follows:

$$U(\pi) - U^* = \sum_{j=1}^{i^*-1} E[X_j^\pi] + (E[X_{i^*}^\pi] - x_{i^*}^\pi) + \sum_{j=i^*+1}^{n} \left(E[X_j^\pi] - \frac{\lambda_j}{\lambda_j + \mu_j} \right) \quad (63)$$

In policy π there is some nonnegative probability that agents $j > i^*$ receive resources and therefore we have that:

$$U(\pi) - U^* \leq \sum_{j=1}^{i^*-1} E[X_j^\pi] + (E[X_{i^*}^\pi] - x_{i^*}^\pi) \tag{64}$$

Furthermore, we have that:

$$E[X_{i^*}^\pi] = \frac{\lambda_i - r\mathbb{P}\left(\mathbf{X} \in \pi_{i^*}\right)}{\lambda_i + \mu_i} \leq \frac{\lambda_i}{\lambda_i + \mu_i} \tag{65}$$

and thus:

$$U(\pi) - U^* \leq \sum_{j=1}^{i^*-1} E[X_j^\pi] + \frac{r - \sum_{j=1}^{i^*} \lambda_j}{\lambda_j + \mu_j} \tag{66}$$

Now we can focus our attention on the first $i^* - 1$ agents. Using Lemma 2 we get:

$$\sum_{j=1}^{i} \mathbb{E}\left[X_i^\pi\right] \leq \frac{2r}{r - \theta_i} \tag{67}$$

where $\theta_i = \sum_{j=1}^{i} \lambda_i$. Let's choose i such that $\theta_i = r - \sqrt{r}$ and we get that:

$$\sum_{j=1}^{i} E[X_j^\pi] \leq 2\sqrt{r} \tag{68}$$

Now let's consider the agents between $i + 1$ and $i^* - 1$. By assumption, we have:

$$\sum_{j=1}^{i^*-1} \lambda_j \leq r \tag{69}$$

$$\sum_{j=1}^{i} \lambda_j + \sum_{j=i+1}^{i^*-1} \lambda_j \leq r \tag{70}$$

and therefore,

$$\sum_{j=i+1}^{i^*-1} \lambda_j \leq \sqrt{r} \tag{71}$$

Because there is a lower bound for λ_j we have there are at most $\frac{\sqrt{r}}{\underline{\lambda}}$ agents between i and i^*. With this, we can conclude that the difference between $U(\pi)$ and U^* is at most given by (11). To get (12) is enough to note that $r < \sum_{i \in N} \lambda_i \leq n\bar{\lambda}$.

□

B.1.3 Proof of Theorem 1

Lemma 9. *Let N be a set of agents and π be a priority queue with order $\succ$ and r a rate of resources. Define the following sequence of instances:*

$$N^k = \{1_1, 1_2, ..., 1_k, ..., n_1, n_2, ..., n_k\} \tag{72}$$

$$r^k = rk \tag{73}$$

$$\tag{74}$$

Let N_k be a sequence of sets of agents parameterized by k that contains k copies of each agent $i \in N$ and $r_k = rk$ is a sequence of resource rates. We define a preference $\succ^k$ by following the original $\succ$ and breaking ties in favor of agents with the smallest subindex:

$$i_j \succ^k i'_{j'} \text{ if } i \succ j \text{ and } i < j \tag{75}$$

Let π_k be a sequence of priority queues based on $\succ^k$. Then we have that:

$$\lim_{k\to\infty} \frac{\sum_{i_j \in N_k} |X_{i_j}^{\pi^k} - x_{i_j}^{\pi^k}|}{k} = 0 \tag{76}$$

Proof. This follows directly from Proposition 2. □

Proof. Consider the following example:

For (3) consider the following example: Choose a small $\delta > 0$ and set $\lambda_1 = 1/\delta$ and $\mu_1 = \delta$. Let N_1 contain agents such that $\lambda_i = 1$ and $\mu_i = 1/\delta$ for all $i \in N_1$. Let N_2 contain agents such that $\lambda_j = 1 - \delta$ and $\mu_i = 1/\delta$. Finally, let N_3 contain agents such that $\lambda_i = 1$ and $\mu_i = \delta$ for all $i \in N_3$. We should N_2 and N_3 such that: $|N_2| = |N_3| = 1/\delta$. Let $r = 1/\delta$.

First, consider the case when $N = \{1\} \cup N_3$. Let π^μ be a priority queue that prioritizes based on vulnerability. Then all of the resources will go to the first agents. The agents in N_2 will be mostly unhoused:

$$\sum_{i\in N} x_i^{\pi^\mu} \approx 1/\delta \tag{77}$$

In a benefit-based priority queue $\pi^{\lambda+\mu}$ most resources would go to agents in N_3:

$$\sum_{i\in N} x_i^{\pi^{\lambda+\mu}} \approx 1 \tag{78}$$

We end with:

$$\sum_{i\in N} x_i^{\pi^\mu} - x_i^{\pi^{\lambda+\mu}} \approx 1/\delta - 1 = |N|(1 - 2/|N|) \tag{79}$$

Now consider the case when $N = N_2 \cup N_3$. Let π^λ be a priority queue that prioritizes based on success. Then all of the resources will go to agents in the group N_2 and none to the other agents.

$$\sum_{i\in N} x_i^{\pi^\lambda} \approx 1/\delta \tag{80}$$

In a benefit-based priority queue $\pi^{\lambda+\mu}$ most resources would go to agents in N_3:

$$\sum_{i \in N} x_i^{\pi^{\lambda+\mu}} \approx \delta \tag{81}$$

We end with:

$$\sum_{i \in N} x_i^{\pi^{\mu}} - x_i^{\pi^{\lambda+\mu}} \approx 1/\delta - 1 = |N| - |N|/2 - 1 = |N|(1/2 - \delta/|N|) \tag{82}$$

To conclude the proof we use Lemma 9 to show that we can create a new instance where the distance between X^π and its fluid approximation is as close as we want.

□

B.1.4 Proof Proposition 2

Proof. We are going to split the set of agents into 3 groups. To make this split we will use 3 sets. In the first group M_1 we will have agents that are likely to get resources. All agents with a priority worse than M_3 are unlikely to receive resources. Finally in M_2 are the agents where is hard to tell how many resources they will get.

$$M_1 = \{i \in N : \sum_{j=1}^{i} = r - r^\alpha\} \tag{83}$$

$$M_3 = \{i \in N : \sum_{j=1}^{i} = r + r^\alpha\} \tag{84}$$

$$M_2 = M_3 \setminus M_1 \tag{85}$$

We can then write the distance as:

$$\sum_{i \in N} |\mathbb{E}\left[X_i^\pi\right] - x_i^\pi| \leq \sum_{i \in M_1} |\mathbb{E}\left[X_i^\pi\right] - x_i^\pi| + \sum_{i \in M_2} |\mathbb{E}\left[X_i^\pi\right] - x_i^\pi| + \sum_{i \in M_3} |\mathbb{E}\left[X_i^\pi\right] - x_i^\pi| \tag{86}$$

For the first sum since $r - r^\alpha < r$ we can use Lemma 2 and we get:

$$\sum_{i \in M_1} |\mathbb{E}\left[X_i^\pi\right] - x_i^\pi| \leq 2r^{1-\alpha} \tag{87}$$

For the third sum, we can directly use Lemma 3 and we get:

$$\sum_{i \in M_3} |\mathbb{E}\left[X_i^\pi\right] - x_i^\pi| \leq \frac{r}{\underline{\mu} + \underline{\lambda}} exp\left\{-\frac{r^{2\alpha}}{8(\bar{\lambda} + \bar{\mu})(r + r^\alpha)(1 + \bar{\mu}/\underline{\lambda})}\right\} \tag{88}$$

For any $i \in M_2$ we know by Proposition 1 and by the definition of x_i^π that both $\mathbb{E}[X_i^\pi], x_i^\pi \in [0, \frac{\lambda_i}{\lambda_i+\mu_i}]$ and thus,

$$\sum_{i \in M_2} |\mathbb{E}[X_i^\pi] - x_i^\pi| \leq \sum_{i \in M_2} \frac{\lambda_i}{\lambda_i + \mu_i} \tag{89}$$

$$\leq \sum_{i \in M_2} \frac{\lambda_i}{\underline{\lambda} + \underline{\mu}} \tag{90}$$

$$= \frac{2r^\alpha}{\underline{\lambda} + \underline{\mu}} \tag{91}$$

Where the last inequality comes from the definition of M_2 and the fact that π is a priority queue. Combining the three expressions we get the statement of the theorem.

□

B.1.5 Proof of other lemmas

Proof of Lemma 2

Proof. We will be using the result from Lemma 1. Define the following Lyapunov function:

$$V(\mathbf{X}) = \sum_{i \in M} X_i \tag{92}$$

This function counts the number of unhoused agents in M and has a maximum change of at most $\nu_{max} = 1$. Let the exception $B = 1$ and the drift be $\gamma = kr - \sum_{i \in M} \lambda_i$. Then for any $\mathbf{X}$ such that $V(\mathbf{X}) \geq B$ we have:

$$GV(\mathbf{X}) = \sum_{i \in M} \lambda_i X_i - r - \sum_{i \in M} \mu_i (1 - X_i) \tag{93}$$

$$\leq -r + \sum_{i \in M} \lambda_i = -\gamma. \tag{94}$$

The inequality comes from setting all X_i's to 1. It is easy to see that when $V(\mathbf{X}) = 0$ everyone is housed and therefore the maximum rate of increase is $f_{max} = \sum_{i \in M} \lambda_i$. By Lemma 1 we get:

$$\mathbb{P}\left(\sum_{i \in M} X_i^\pi \geq 1 + 2k\right) \leq \left(\frac{\sum_{i \in M} \lambda_i}{\sum_{i \in M} \lambda_i + r - \sum_{i \in M} \lambda_i}\right)^k = \left(\frac{\sum_{i \in M} \lambda_i}{r}\right)^k \tag{95}$$

Using the fact that for any non-negative integer-valued random variable X, $E[X] = \sum_{i=0}^\infty \mathbb{P}(X > i)$ and that $P(X > 2k) + P(X > 2k + 1) \leq 2P(X > 2k)$ we get:

$$\sum_{i \in M} E[X_i^\pi] \leq 2 \sum_{k=0}^{|M|} \left(\frac{\sum_{i \in M} \lambda_i}{r}\right)^k \tag{96}$$

$$\leq \frac{2r}{r - \sum_{i \in M} \lambda_i} \tag{97}$$

where the second inequality comes from calculating an infinite sum rather than the first $|M|$ terms.

□

Proof of Lemma 3

Proof. We will first calculate the probability that an agent not in M will receive a resource. This is equivalent to:

$$\mathbb{P}\left(\mathbf{X}^\pi \in \cup_{i \in N \setminus M} \pi_i\right) = \mathbb{P}\left(\mathbf{X}^\pi \in \{\mathbf{X} : X_i = 0, \forall i \in M\}\right) \tag{98}$$

$$= \mathbb{P}\left(\sum_{i \in M} X_i^\pi = 0\right) \tag{99}$$

We will once again invoke Lemma 1. Define the following Lyapunov function:

$$V(\mathbf{X}) = |M| - \sum_{i \in M} X_i \tag{100}$$

This function counts the number of housed agents in M and has a maximum change of at most $\nu_{max} = 1$. By definition $r^\alpha = \sum_{i \in M} \lambda_i - r$ and let $B = |M| - \frac{r^\alpha}{2(\bar{\lambda}+\bar{\mu})}$ and the drift be $\gamma = r^\alpha/2$. For any $\mathbf{X}$ the rate of change of the Lyapunov function is:

$$GV(\mathbf{X}) = r + \sum_{i \in M} X_i \mu_i - \sum_{i \in M} (1 - x_j)\lambda_i \leq -r^\alpha + y(\bar{\lambda} + \bar{\mu}) \tag{101}$$

where $y = \sum_{i \in M} x_i$. For the cases where $V(\mathbf{X}) \geq B$ we have that $y \leq \frac{r^\alpha}{2(\bar{\lambda}+\bar{\mu})}$ and thus:

$$GV(\mathbf{X}) \leq -r^\alpha + y(\bar{\lambda} + \bar{\mu}) \leq -\frac{r^\alpha}{2} = -\gamma \tag{102}$$

The maximum rate of increase of this function is $f_{max} \leq |i|\bar{\mu} + r$ when everyone is unhoused. Invoking Lemma 1 we get:

$$\mathbb{P}\left(|M| - \sum_{i \in M} X_j^\pi \geq |M| - \frac{r^\alpha}{2(\bar{\lambda}+\bar{\mu})} + 2k\right) \leq \left(\frac{|M|\bar{\mu} + r}{|M|\bar{\mu} + r + r^\alpha/2}\right)^k \tag{103}$$

We are going to set $k = \frac{r^\alpha}{4(\bar{\lambda}+\bar{\mu})}$ which is possible since by assumption $r^\alpha \geq 4(\bar{\lambda} + \bar{\mu})$:

$$\mathbb{P}\left(\sum_{i \in M} X_i^\pi \leq 0\right) \leq \left(\frac{|M|\bar{\mu} + r}{|M|\bar{\mu} + r + r^\alpha/2}\right)^{\frac{r^\alpha}{4(\bar{\lambda}+\bar{\mu})}} \tag{104}$$

Note that

$$\sum_{i \in N \setminus M} |\mathbb{E}[X_i^\pi] - x_i^\pi| = \sum_{i \in N \setminus M} \frac{r_i^\pi}{\lambda_i + \mu_i} \tag{105}$$

$$\leq \frac{1}{\underline{\mu} + \underline{\lambda}} \sum_{i \in N \setminus M} r_i^\pi \tag{106}$$

Given the definition of r_i^π and (104) we get:

$$\sum_{i\in N\setminus M} |\mathbb{E}\left[X_i^\pi\right] - x_i^\pi| \leq \frac{r}{\underline{\mu}+\underline{\lambda}}\left(\frac{|M|\bar{\mu}+r}{|M|\bar{\mu}+r+r^\alpha/2}\right)^{\frac{r^\alpha}{4(\bar{\lambda}+\bar{\mu})}} \tag{107}$$

$$\leq \frac{r}{\underline{\mu}+\underline{\lambda}} exp\left\{-\frac{r^{2\alpha}}{8(\bar{\lambda}+\bar{\mu})(|M|\bar{\mu}+r+r^\alpha/2)}\right\} \tag{108}$$

$$\leq \frac{r}{\underline{\mu}+\underline{\lambda}} exp\left\{-\frac{r^{2\alpha}}{8(\bar{\lambda}+\bar{\mu})((r+r^\alpha)\bar{\mu}/\underline{\lambda}+r+r^\alpha/2)}\right\} \tag{109}$$

$$\leq \frac{r}{\underline{\mu}+\underline{\lambda}} exp\left\{-\frac{r^{2\alpha}}{8(\bar{\lambda}+\bar{\mu})(r+r^\alpha)(1+\bar{\mu}/\underline{\lambda})}\right\} \tag{110}$$

Where the second inequality comes from the known property $(1-x)^k \leq e^{-kx}$, the third one comes from noting that $|M|\underline{\lambda} \leq \sum_{i\in M}\lambda_i = r + r^\alpha$ and the last one comes from simplifying the expression. □

B.2 Prioritization based on waiting time: proofs and lemmas

B.2.1 Preliminaries

Let $Z_\mu \sim Exp(\mu)$ be the time an agent with vulnerability μ will take to find housing without help and $p(\mu, \mathbf{F})$ denote the probability that the policymaker helps this agent under intervention $\mathbf{F}$:

$$p(\mu, \mathbf{F}) = \mathbb{P}_{Y\sim \mathbf{F}}(Y \leq Z_\mu) \tag{111}$$

Let $w(\mu, \mathbf{F})$ be the expected waiting time of an agent under intervention $\mathbf{F}$.

$$w(\mu, \mathbf{F}) = \mathbb{E}_{Y\sim \mathbf{F}}\left[\min\{Z_\mu, Y\}\right] \tag{112}$$

These two are related by the following equation.

Lemma 10. *For any μ and $\mathbf{F}$ we have that $w(\mu, \mathbf{F}) = \frac{1-p(\mu,\mathbf{F})}{\mu}$.*

Proof. Let $Z_\mu \sim Exp(\mu)$. First, note that w can be written as:

$$w(\mu, F) = E_{Y\sim F}[\min\{Y, Z_\mu\}] = \int_0^\infty P_{Y\sim F}(Y > t)e^{-\mu t}dt \tag{113}$$

Also, $1 - p(\mu, F)$ can be calculated using the marginal distribution:

$$1 - p = \int_0^\infty P_{Y\sim F}(Y > t)\mu e^{-\mu t}dt = \mu w(\mu, F) \tag{114}$$

From which we can conlcude that $w(\mu, F) = \frac{1-p(\mu,F)}{\mu}$. □

Lemma 11. *For any intervention $\mathbf{F}$ on agent i the expected unhoused time and rate at which the* *policymakers use resources on this agent is given by:*

$$x_i^F = \frac{\lambda_i(1-p(\mu_i,\mathbf{F}))}{\mu_i+\lambda_i(1-p(\mu_i,\mathbf{F}))} \tag{115}$$

$$r_i^F = \frac{\lambda_i\mu_i p(\mu_i,\mathbf{F})}{\mu_i+\lambda_i(1-p(\mu_i,\mathbf{F}))} = \lambda_i p(\mu_i,\mathbf{F})(1-x_i^F) \tag{116}$$

Proof. Let $X \sim \mathbf{F}, Y_{\mu_i} \sim Exp(\mu_i)$. Also, Let $u_i^F \sim \min\{Y_{\mu_i}, X\}$ be a random variable that denotes the time agent i spends unhoused under intervention $\mathbf{F}$ and $h_i \sim Exp(\lambda_i)$ denote the time agent i spends housed (note that this time does not depend on the policy). Using the renewal reward theorem we can calculate the expected time spent housed as follows:

$$1-x_i^F = \lim_{T\to\infty}\frac{H_i(T)}{T} = \frac{E[h_i]}{E[h_i]+E[u_i^F]} = \frac{1/\lambda_i}{1/\lambda_i+w(\mu_i,\mathbf{F})} = \frac{1}{1+\lambda_i w(\mu_i,\mathbf{F})} \tag{117}$$

Let ρ_i be a random variable that represents the probability of getting help from the designer once agents i becomes unhoused. The expected renewal time of process $N_i(t)$ is:

$$\nu_i = E[\rho_i(h_i+u_i^F)] = E[\rho_i](E[h_i]+E[u_i^F]) = \frac{1}{p(\mu_i,\mathbf{F})}\left(\frac{1}{\lambda_i}+w(\mu_i,\mathbf{F})\right) \tag{118}$$

The independence of the random variables was heavily used in the previous equation. We can use the elementary renewal theorem and get:

$$r_i^F = \lim_{T\to\infty}\frac{N_i(T)}{T} = \frac{1}{\nu_i} = \frac{p(\mu_i,F)}{\frac{1}{\lambda_i}+w(\mu_i,\mathbf{F})} \tag{119}$$

Using Lemma 10 we can simplify this to:

$$r_i^F = \frac{\lambda_i\mu_i p(\mu_i,\mathbf{F})}{\mu_i+\lambda_i(1-p(\mu_i,\mathbf{F}))} = (1-x_i)^F\lambda_i p(\mu_i,\mathbf{F}) \tag{120}$$

□

Corollary 1. *We can write the unhoused time of agent i based on resources used on this agent as follows:*

$$x_i^F = \frac{\lambda_i - r_i^F}{\lambda_i+\mu_i} \tag{121}$$

We can use Lemma 11 and get the rate of resources used by agent i under a *FIFO Queue* $\mathbf{F}_t$:

$$r_i^{F_t} = \frac{\mu_i\lambda_i e^{-\mu_i t}}{\mu_i+\lambda_i(1-e^{-\mu_i t})} \tag{122}$$

We can use Lemma 11 and get the rate of resources used by agent i under a *LIFO Queue* $\mathbf{L}_q$:

$$r_i^{L_q} = \frac{\mu_i\lambda_i q}{\mu_i+\lambda_i(1-q)} \tag{123}$$

Lemma 12. *For a fixed intervention $\mathbf{F}$ the function $g(\mu) = p(\mu,\mathbf{F})^{1/\mu}$ is nondecreasing in μ.*

Proof. Note that if $X \sim \mathbf{F}$ then $p(\mu,\mathbf{F}) = \mathbb{E}\left[e^{-\mu X}\right]$. Now take $\mu_i < \mu_j$ and define $a = \frac{\mu_j}{\mu_i}$ and $b = \frac{\mu_j}{\mu_j-\mu_i}$. Note that $\frac{1}{a}+\frac{1}{b} = 1$ and therefore we can use Holder's inequality:

$$\left(\mathbb{E}\left[(e^{-\mu_i X})^a\right]\right)^{1/a}\left(\mathbb{E}\left[1^b\right]\right)^{1/b} \geq \mathbb{E}\left[e^{-\mu_i X}\right] \tag{124}$$

Replacing a and b and taking root of μ_i we get:

$$\left(\mathbb{E}\left[e^{-\mu_j X}\right]\right)^{1/\mu_j} \geq \left(\mathbb{E}\left[e^{-\mu_i X}\right]\right)^{1/\mu_i} \tag{125}$$

$$p(\mu_j, \mathbf{F})^{1/\mu_j} \geq p(\mu_i, \mathbf{F})^{1/\mu_i} \tag{126}$$

□

Lemma 13. *For all i, the rate of resources $r_i^{F_t}$ of FIFO queue is continuous and decreasing in t for $t > 0$.*

Proof. This follows directly from (122) and Lemma 11. □

Lemma 14. *For a fixed $\mathbf{F}$ the probability of getting help $p(\mu, \mathbf{F})$ is nonincreasing in μ.*

Proof. Let f be the PDF of $\mathbf{F}$. Then we can write $p(\mu, \mathbf{F})$ as follows:

$$p(\mu, \mathbf{F}) = \int_0^\infty f(x) e^{-\mu x} dx \tag{127}$$

If we take the derivative with respect to μ we get:

$$\frac{\partial p(\mu, \mathbf{F})}{\partial \mu} = -\int_0^\infty x f(x) e^{-\mu x} dx \leq 0 \tag{128}$$

where the inequality comes from the fact that $x, f(x), e^{-\mu x} \geq 0$. □

Lemma 15. *For all i, the rate of resource $r_i^{A_q}$ for LIFO Queues is continuous and increasing in q on the interval $[0, 1]$.*

Proof. This follows from (123) and Lemma 11. □

B.2.2 Proof of Lemma 5

Proof. We will show that there is a *FIFO Queue* $\mathbf{F}_{t_k}$ such that $\mathbf{F}_{t_k} \succeq^k \mathbf{A}$ and $r^{F_{t_k}} \leq r^A$ for all $k \in N$. We will do this using induction. First, for the base case let $t_1 = -\frac{log(p(\mu_1, \mathbf{A}))}{\mu_1}$ and note that:

$$p(\mu_1, \mathbf{F}_{t_1}) = p(\mu_1, \mathbf{A}) \tag{129}$$

$$p(\mu_k, \mathbf{F}_{t_1}) = e^{-t_1 \mu_k} = p(\mu_1, \mathbf{A})^{\mu_k/\mu_1} \leq p(\mu_k, \mathbf{A}), \quad k > 1 \tag{130}$$

Where the last inequality comes from Lemma 12. From Lemma 11 we know that $r_1^{F_{t_1}} = r_1^A$ and $r_k^{F_{t_1}} \leq r_k^A$ for $k > 1$. Therefore $r^{F_{t_1}} \leq r^F$ and $\mathbf{F}_{t_1} \succeq^1 \mathbf{A}$. We proceed with the following induction hypothesis: there is a t_i such that $r^{F_{t_i}} \leq r^A$ and $\mathbf{F}_{t_i} \succeq^i \mathbf{A}$. And we want to show that there is a t_{i+1} such that $r^{F_{t_{i+1}}} \leq r^A$ and $\mathbf{F}_{t_{i+1}} \succeq^{i+1} \mathbf{A}$. If $\sum_{j=1}^{i+1} r_j^{F_{t_i}} \geq \sum_{j=1}^{i+1} r_j^A$ then we set $t_{i+1} = t_i$ and we are done. If $\sum_{j=1}^{i+1} r_j^{F_{t_i}} < \sum_{j=1}^{i+1} r_j^A$ then set t_{i+1} such that:

$$\sum_{j=1}^{i+1} r_j^{F_{t_{i+1}}} = \sum_{j=1}^{i+1} r_j^A \tag{131}$$

Note that $t_{i+1} < t_i$ and by Lemma 13 we know that all $r_j^{F_t}$ are decreasing in t and therefore we have that:

$$\sum_{j=1}^{k} r_j^{F_{t_{i+1}}} > \sum_{j=1}^{k} r_j^{F_{t_i}} \geq \sum_{j=1}^{k} r_j^{A}, \quad 1 \leq k \leq i \tag{132}$$

and therefore $\mathbf{F}_{t_{i+1}} \succeq^{i+1} \mathbf{A}$. Combining (131) and (132) we get $r_{i+1}^{F_{t_{i+1}}} < r_{i+1}^{A}$ and therefore $p(\mu_{i+1}, \mathbf{F}_{t_{i+1}}) \leq p(\mu_{i+1}, \mathbf{A})$. From this we conclude that $t_{i+1} \geq -\frac{log(p(\mu_{i+1}, \mathbf{A}))}{\mu_{i+1}}$. Thus:

$$p(\mu_{i+1}, \mathbf{F}_{t_{i+1}}) = e^{-\mu_{i+1} t_{i+1}} \leq p(\mu_{i+1}, \mathbf{A}) \tag{133}$$

$$p(\mu_k, \mathbf{F}_{t_{i+1}}) = e^{-\mu_k t_{i+1}} \leq p(\mu_{i+1}, \mathbf{A})^{\mu_k/\mu_{i+1}} \leq p(\mu_k, \mathbf{A}), \quad k > i+1 \tag{134}$$

where the last inequality of (134) comes from Lemma 12. Combining (134) with (131) and Lemma 11 we get that $r^{F_{t_{i+1}}} \leq r^A$. Note that for t_n we necessarily will have $r^{F_{t_n}} = r^A$.

□

B.2.3 Proof of Lemma 6

Proof. We will show that there is an *LIFO Queue* $\mathbf{L}_{q_k}$ such that $\mathbf{A} \succeq^k \mathbf{L}_{q_k}$ and $r^{L_{q_k}} \geq r^A$ for all $k \in N$. We will do this using induction. First, for the base case let $q_1 = p(\mu_1, \mathbf{A})$ and note that:

$$p(\mu_1, \mathbf{L}_{q_1}) = q_1 = p(\mu_1, \mathbf{A}) \tag{135}$$

$$p(\mu_k, \mathbf{L}_{q_1}) = q_1 \geq p(\mu_k, \mathbf{A}), \quad k > 1 \tag{136}$$

Where the last inequality comes from Lemma 14 and the assumption that the μ_i are in increasing order. From Lemma 11 we know that $r_1^{L_{q_1}} = r_1^A$ and $r_k^{L_{q_1}} \geq r_k^A$ for $k > 1$ and therefore $r^{L_{q_1}} \geq r^A$ and $\mathbf{A} \succeq^1 \mathbf{L}_{q_1}$. We proceed with the following induction hypothesis: there is a q_i such that $r^{L_{q_i}} \geq r^A$ and $\mathbf{A} \succeq^i \mathbf{L}_{q_i}$. And we want to show that there is a q_{i+1} such that $r^{L_{q_{i+1}}} \geq r^A$ and $\mathbf{A} \succeq^{i+1} \mathbf{L}_{q_{i+1}}$. If $\sum_{j=1}^{i+1} r_j^{L_{q_i}} \leq \sum_{j=1}^{i+1} r_j^A$ then we set $q_{i+1} = q_i$ and we are done. If $\sum_{j=1}^{i+1} r_j^{L_{q_i}} > \sum_{j=1}^{i+1} r_j^A$ then set q_{i+1} such that:

$$\sum_{j=1}^{i+1} r_j^{L_{q_{i+1}}} = \sum_{j=1}^{i+1} r_j^A \tag{137}$$

Note that $q_{i+1} < q_i$ and by Lemma 15 we have that.

$$\sum_{j=1}^{k} r_j^{L_{q_{i+1}}} < \sum_{j=1}^{k} r_j^{L_{q_i}} \leq \sum_{j=1}^{k} r_j^{A}, \quad 1 \leq k \leq i \tag{138}$$

where the last inequality comes from the inductive hypothesis. Therefore $\mathbf{A} \succeq^{i+1} \mathbf{L}_{q_{i+1}}$. Combining (137) and (138) we get $r_{i+1}^{L_{q_{i+1}}} \geq r_{i+1}^A$ and therefore $p(\mu_{i+1}, \mathbf{L}_{q_{i+1}}) \geq p(\mu_{i+1}, \mathbf{A})$. From this we conclude that $q_{i+1} \geq p(\mu_{i+1}, \mathbf{A})$. Thus:

$$p(\mu_{i+1}, \mathbf{L}_{q_{i+1}}) = q_{i+1} \geq p(\mu_{i+1}, \mathbf{A}) \tag{139}$$

$$p(\mu_k, \mathbf{L}_{q_{i+1}}) = q_{i+1} \geq p(\mu_{i+1}, \mathbf{A}), \quad k > i+1 \tag{140}$$

where the last inequality of (140) comes from Lemma 15. Combining (140) with (137) and Lemma 11 we get that $r^{L_{q_{i+1}}} \geq r^A$. Note that for q_n we necessarily will have $r^{L_{q_n}} = r^A$. □

B.2.4 Proof of Lemma 4

Proof. Define

$$\delta_i = r_i^F - r_i^G, \tag{141}$$

$$\gamma_i = \frac{1}{\lambda_i + \mu_i}, \tag{142}$$

$$s_i = \sum_{j=1}^{i} \delta_j, \tag{143}$$

$$d_i = \gamma_i - \gamma_{i+1}. \tag{144}$$

Then by Corollary 1 we can write the difference in unhoused population as:

$$U(\mathbf{G}) - U(\mathbf{F}) = \sum_{i=1}^{n} x_i^F - x_i^G = \sum_{i=1}^{n} \frac{r_i^F - r_i^G}{\lambda_i + \mu_i} = \sum_{i=1}^{n} \delta_i \gamma_i = \sum_{i=1}^{n} s_i d_i. \tag{145}$$

By definition of majorization we know that $s_i \geq 0$. By assumption, $\lambda_i + \mu_i \leq \lambda_j + \mu_j$ for $j \geq i$, so $d_i \geq 0$. Therefore, the difference in (145) is non-negative, as claimed. □

B.2.5 Proof of Theorem 3

Proof. This follows immediately from Lemma 4, Lemma 5, and Lemma 6. □

B.2.6 Proof of Theorem 5

This theorem follows from Lemma 16.

Lemma 16. *Let* $\mathbf{F}$ *and* $\mathbf{G}$ *be two anonymous interventions and suppose that* $\mu_i = \mu$ *for all agents* $i \in N$*. Then* $r^F > r^G$ *if and only if* $x_i^F < x_i^G$ *for all agents* i*.*

Proof. First suppose $r^F > r^G$. Then there must exist some i such that $r_i^F > r_i^G$. From Lemma 11 we get:

$$r_i^F = \frac{\mu \lambda_i p(\mu, \mathbf{F})}{\mu + \lambda_i(1 - p(\mu, \mathbf{F}))} \tag{146}$$

From Lemma 11 we know that r_i^F is increasing in $p(\mu, \mathbf{F})$ and therefore, $p(\mu, \mathbf{F}) > p(\mu, \mathbf{G})$. Because $p(\mu, \mathbf{F})$ is the same for all agents under policy $\mathbf{F}$ and $p(\mu, \mathbf{G})$ is the same for all agents under policy $\mathbf{G}$ we can use Lemma 11 again and conclude that: $x_i^F < x_i^G$ for all agents i.

Now suppose $x_i^F < x_i^G$ for all agents i then we can again conclude from Lemma 11 that $p(\mu, \mathbf{F}) > p(\mu, \mathbf{G})$. From this, it follows naturally that $r_i^F > r_i^G$ and therefore $r^F > r^G$. □

B.2.7 Menus lemmas and proofs

Lemma 17. *Adding intervention F to menu $\mathcal{M}$ weakly increases the number of resources used: $r^{\mathcal{M}\cup\{F\}} \geq r^{\mathcal{M}}$.*

Proof. This is straightforward. □

Lemma 18. *Let $\mathcal{M} = \{F^{(1)}, F^{(2)}, ...\}$ be a menu of interventions and F be an single intervention. If for some $F^{(i)}$ we have that $r^{F^{(i)}} > r^F$, then $r^{\mathcal{M}} > r^F$.*

Proof. Start with a menu $\mathcal{M}' = \{F^{(i)}\}$ and add interventions until you get $\mathcal{M}$. By Lemma 17 we get that $r^{\mathcal{M}} > r^F$. □

Proof of Lemma 7

Proof. Let $\mathbf{A}_{(i)} = c_i(\mathcal{M})$ be the intervention selected by agent i. We will use induction to show that for every $i \in N$ there is a *FIFO Queue* $\mathbf{F}_{t_i}$ such that:

$$\mathbf{F}_{t_i} \succeq^i \mathcal{M} \tag{147}$$

$$\sum_{j=1}^{i} r_j^{F_{t_i}} = \sum_{j=1}^{i} r_j^{A_{(j)}} \tag{148}$$

$$r_k^{F_{t_i}} \leq r_k^{A_{(k)}}, \quad \forall k > i \tag{149}$$

For the base case set $t_1 = -log(p(\mu_1, A_{(1)}))/\mu_1$. First note that:

$$p(\mu_1, \mathbf{F}_{t_1}) = e^{-\mu_1 t_1} = p(\mu_1, \mathbf{A}_{(1)}) \tag{150}$$

Therefore, $\mathbf{F}_{t_1} \succeq^1 \mathbf{A}_{(1)}$. For any agent $i > 1$ we have:

$$p(\mu_i, \mathbf{F}_{t_1}) = e^{-\mu_i t_1} = p(\mu_1, \mathbf{A}_{(1)})^{\frac{\mu_i}{\mu_1}} \leq p(\mu_i, \mathbf{A}_{(1)}) \leq p(\mu_i, \mathbf{A}_{(i)}) \tag{151}$$

where the first inequality comes from Lemma 12 and the fact that $\mu_1 \leq \mu_i$. The final inequality becomes an equality if agent i chooses the same option as agent 1, ie $c_i(\mathcal{M}) = c_1(\mathcal{M})$. From Lemma 11 and (151) we can conclude that $r_i^{F_{t_1}} \leq r_i^{A_{(i)}}$ for all $i > 1$, and therefore $r^{F_{t_1}} \leq r^{\mathcal{M}}$.

We proceed with the following inductive hypothesis: for agent i there is a *FIFO Queue* $\mathbf{F}_{t_i}$ such that equations (147), (148) and (149) hold and want to show that the same conditions hold for $i+1$. First, if $r_{i+1}^{F_{t_i}} = r_{i+1}^{A_{(i+1)}}$ then set $t_{i+1} = t_i$ and we are done. By the inductive hipothesis we know that $r_{i+1}^{F_{t_i}} > r_{i+1}^{A_{(i+1)}}$ is not possible. For $r_{i+1}^{F_{t_i}} < r_{i+1}^{A_{(i+1)}}$ set t_{i+1} such that:

$$\sum_{j=1}^{i+1} r_j^{F_{t_{i+1}}} = \sum_{j=1}^{i+1} r_j^{A_{(j)}} \tag{152}$$

From the monotonicity of *FIFO Queues* shown in Lemma 15 we have that $t_{i+1} < t_i$ and therefore:

$$\sum_{j=1}^{k} r_j^{F_{t_{i+1}}} > \sum_{j=1}^{k} r_j^{F_{t_i}} \geq \sum_{j=1}^{k} r_j^{A_{(j)}}, \quad 1 \leq k \leq i \tag{153}$$

and combining this with (152) we get that $\mathbf{F}_{t_{i+1}} \succeq^{i+1} \mathcal{M}$. Combining (152) and (153) we get that $r_{i+1}^{F_{t_{i+1}}} < r_{i+1}^{A_{(i+1)}}$. Combining this with Lemma 11 we get:

$$p(\mu_{i+1}, \mathbf{F}_{t_{i+1}}) < p(\mu_{i+1}, \mathbf{A}_{(i+1)}) \tag{154}$$

Because $p(\mu_{i+1}, \mathbf{F}_{t_{i+1}}) = e^{-\mu_{i+1}t_{i+1}}$ we have that $t_{i+1} > -log(p(\mu_{i+1}, A_{(i+1)}))/\mu_{i+1}$. Therefore, for any $k > i+1$ we have:

$$p(\mu_k, \mathbf{F}_{t_{i+1}}) = p(\mu_{i+1}, \mathbf{A}_{(i+1)})^{\frac{\mu_k}{\mu_{i+1}}} \leq p(\mu_k, \mathbf{A}_{(i+1)}) \leq p(\mu_k, A_{(k)}) \tag{155}$$

Where again the first inequality comes from Lemma 12 and the fact that $\mu_k \geq \mu_{i+1}$. The final inequality becomes an equality if $c_k(\mathcal{M}) = c_{i+1}(\mathcal{M})$. We conclude the proof by invoking Lemma 11 to conclude that $r_k^{F_{t_{i+1}}} \leq r_k^{A_{(k)}}$ for all $k > i+1$. We have shown that (149) holds for $i+1$.

□

Proof of Lemma 8

Proof. Let $\mathbf{A}_{(i)} = c_i(\mathcal{M})$ be the intervention selected by agent i. We will use induction to show that for every $i \in N$ there is a *LIFO Queue* $\mathbf{L}_{q_i}$ such that

$$\mathcal{M} \succeq^i \mathbf{L}_{q_i} \tag{156}$$

$$\sum_{j=1}^{i} r_j^{L_{q_i}} = \sum_{j=1}^{i} r_j^{A_{(j)}} \tag{157}$$

$$r_k^{L_{q_i}} \geq r_k^{A_{(k)}}, \quad \forall k > i \tag{158}$$

For the base case set $q_1 = p(\mu_1, A_{(1)})$. First note that:

$$p(\mu_1, \mathbf{L}_{q_1}) = q_1 = p(\mu_1, \mathbf{A}_{(1)}) \tag{159}$$

Therefore, $\mathcal{M} \succeq^i \mathbf{L}_{q_1}$. For any agent $i > 1$ we have:

$$p(\mu_i, \mathbf{L}_{q_1}) = q_1 = p(\mu_1, \mathbf{A}_{(1)}) \geq p(\mu_1, \mathbf{A}_{(i)}) \geq p(\mu_i, \mathbf{A}_{(i)}) \tag{160}$$

where the inequality comes from the fact that agent i chooses optimally (equation (35)) and the fact that $\mu_1 \leq \mu_i$. From Lemma 11 and (160) we can conclude that $r_i^{L_{q_1}} \geq r_i^{A_{(i)}}$ for all $i > 1$, and therefore $r^{L_{q_1}} \geq r^{\mathcal{M}}$. We proceed with the following inductive hypothesis: for agent i there is a *LIFO Queue* $\mathbf{L}_{q_i}$ such that equations (156), (157) and (158) hold and want to show that the same conditions hold for $i+1$. First, if $r_{i+1}^{L_{t_i}} = r_{i+1}^{A_{(i+1)}}$ then set $q_{i+1} = q_i$ and we are done. By inductive hypothesis the case $r_{i+1}^{L_{q_i}} < r_{i+1}^{A_{(i+1)}}$ is not possible. Now if $r_{i+1}^{L_{q_i}} > r_{i+1}^{A_{(i+1)}}$ then set q_{i+1} such that:

$$\sum_{j=1}^{i+1} r_j^{L_{q_{i+1}}} = \sum_{j=1}^{i+1} r_j^{A_{(j)}} \tag{161}$$

From Lemma 15 we have that $q_{i+1} < q_i$ and therefore:

$$\sum_{j=1}^{k} r_j^{L_{q_{i+1}}} < \sum_{j=1}^{k} r_j^{L_{q_i}} \leq \sum_{j=1}^{k} r_j^{A_{(j)}}, \quad 1 \leq k \leq i \tag{162}$$

and combining this with (161) we get that $\mathcal{M} \succeq^{i+1} \mathbf{L}_{q_{i+1}}$. Combining (161) and (162) we get that $r_{i+1}^{L_{q_{i+1}}} > r_{i+1}^{A_{(i+1)}}$. Combining this with Lemma 11 we get:

$$p(\mu_{i+1}, \mathbf{L}_{q_{i+1}}) > p(\mu_{i+1}, \mathbf{A}_{(i+1)}) \tag{163}$$

Because $p(\mu_{i+1}, \mathbf{L}_{q_{i+1}}) = q_{i+1}$ we have that $q_{i+1} \geq p(\mu_{i+1}, \mathbf{A}_{(i+1)})$ and therefore for any $k > i+1$:

$$p(\mu_k, \mathbf{L}_{q_{i+1}}) = q_{i+1} \geq p(\mu_{i+1}, \mathbf{A}_{(i+1)}) \geq p(\mu_k, \mathbf{A}_{(k)}) \tag{164}$$

Where again the final inequality comes from the fact that agent $k \geq i+1$ chooses optimally and the fact that $\mu_k \geq \mu_{i+1}$. We conclude the proof by invoking Lemma 11 to conclude that $r_k^{L_{q_{i+1}}} \geq r_k^{A_{(k)}}$ for all $k > i+1$. We showed that (158) holds for $i+1$.

□

Proof of Theorem 4

Proof. This follows from noting that a Menu $\mathcal{M}$ induces an intervention $\mathbf{F}$ (which is not necessarily anonymous). We then combine Lemma 4, Lemma 7 and Lemma 8 to get the result. □